Product Awareness, Industry Life Cycles, and Aggregate Profits
Online Technical Appendix

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Appendix A Fully Dynamic Model and Aggregation

All references to the Main paper are prefixed by "Main". Define the partial derivative with respect to a as the operator $\boldsymbol{\partial}_a$.

A.1 Dynamic Equilibrium

See Main Appendix B.3 for a derivation of the normalization and law of motion of the age distribution. Given an initial condition for the economy (i.e., $k(0)$, $M(0)$, $\Phi(0,a)$), the following characterize the transition dynamics:

Proposition 1 (Dynamic Equilibrium)**.** *When $i_M(t)$ is non-binding, the equilibrium relationship between $\hat{M} \equiv M(t)/z_M(t)$ and $k(t)$ solves the following implicit equation,*

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z(t)Q(t)B(t)^{-1} z_M(t)^{\frac{1}{\kappa-1}} k^{\alpha} \hat{M}^{\frac{1}{\kappa-1}} \left(\frac{1}{\kappa-1}\hat{M}^{-1} - \alpha k^{-1}\right) \tag{A.1}$$

Define a function, $\zeta(\cdot)$ as the solution to this implicit equation such that $\hat{M} = \zeta(t,k)$. Then given $z(t)$ and $z_M(t)$, the following system of ODEs in $C(t)$ and $k(t)$ characterize the equilibrium

$$\boldsymbol{\partial}_t k(t) = \frac{z(t)Q(t)B(t)^{-1}k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t))\zeta(t,k(t)) - \boldsymbol{\partial}_t \zeta(t,k(t))}{1 + \boldsymbol{\partial}_k \zeta(t,k(t))} \tag{A.2}$$

$$\boldsymbol{\partial}_t \log C(t) = \frac{1}{\gamma}\left(\alpha z(t)Q(t)B(t)^{-1}k(t)^{\alpha-1} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} - \rho - \delta_k\right) \tag{A.3}$$

Where $Q(t)$ and $B(t)$ are a function of $\Phi(t,a)$, and given the equilibrium $M(t)$, $\Phi(t,a)$ evolves according to,

$$\boldsymbol{\partial}_t \Phi(t,a) = -\boldsymbol{\partial}_a \Phi(t,a) + (1-\Phi(t,a))\left(\boldsymbol{\partial}_t \log z_M(t) + \frac{\boldsymbol{\partial}_t k(t) \boldsymbol{\partial}_k \zeta(t,k(t)) + \boldsymbol{\partial}_t \zeta(t,k(t))}{\zeta(t,k(t))} + \delta_M\right) \tag{A.4}$$

Proof. From Main (29) and (B.27) with inelastic labor supply normalized to 1, define a $b(t)$ to simplify the algebra,

$$b(t) \equiv \frac{\left[\mathbb{E}_t\left[(1-f_0(a))\Upsilon(a)^{1-\kappa}q(a)\right]\right]^{1/\varsigma}}{\mathbb{E}_t\left[(1-f_0(a))\Upsilon(a)^{-\kappa}q(a)\right]} = \frac{Q(t)}{B(t)} \tag{A.5}$$

$$Y(t) = z(t)b(t)k(t)^{\alpha}M(t)^{\frac{1}{\kappa-1}} \tag{A.6}$$

With this, define a present-value Hamiltonian for Main (36) as,

$$\mathcal{H} \equiv \frac{1}{1-\gamma}\left[\underbrace{\overbrace{z(t)b(t)M(t)^{\frac{1}{\kappa-1}}k(t)^{\alpha}}^{\equiv f(t,k,M)} - i_k(t) - i_M(t)}_{\equiv C(t)}\right]^{1-\gamma}$$
$$+ \lambda_k(t)\left(-\delta_K k(t) + i_k(t)\right) + \lambda_M(t)\left(-\delta_M M(t) + z_M(t)i_M(t)\right) + \lambda_i i_M(t) \tag{A.7}$$

where $\lambda_k(t)$, $\lambda_M(t)$, and $\lambda_i(t)$ are co-state variables, with the complementary condition $\lambda_i(t)i_M(t) = 0$ and $i_M(t) \geq 0$. With the $\boldsymbol{\partial}_{i_k}\mathcal{H} = 0$ first-order necessary condition,

$$C(t)^{-\gamma} = \lambda_k(t) \tag{A.8}$$

Differentiate, divide by $\lambda_k(t)$, and reorganize,

$$\frac{\boldsymbol{\partial}_t\lambda_k(t)}{\lambda_k(t)} = -\gamma\frac{\boldsymbol{\partial}_t C(t)}{C(t)} = -\gamma\boldsymbol{\partial}_t \log C(t) \tag{A.9}$$

For the $\boldsymbol{\partial}_{i_M}\mathcal{H} = 0$ first-order necessary and complementarity conditions,

$$C(t)^{-\gamma} \geq z_M(t)\lambda_M(t) \tag{A.10}$$

$$= z_M(t)\lambda_M(t), \text{ if } i_M(t) > 0 \tag{A.11}$$

If the constraint is non-binding at t, then use (A.8) and (A.11), differentiate, and divide by $z_M(t)\lambda_M(t)$ to find

$$\frac{\boldsymbol{\partial}_t\lambda_M(t)}{\lambda_M(t)} = \frac{\boldsymbol{\partial}_t\lambda_k(t)}{\lambda_k(t)} - \frac{\boldsymbol{\partial}_t z_M(t)}{z_M(t)} \tag{A.12}$$

For the $\boldsymbol{\partial}_k\mathcal{H} = \rho\lambda_k(t) - \boldsymbol{\partial}_t\lambda_k(t)$ and $\boldsymbol{\partial}_M\mathcal{H} = \rho\lambda_M(t) - \boldsymbol{\partial}_t\lambda_M(t)$ first-order conditions for the present value Hamiltonian

$$C(t)^{-\gamma}\boldsymbol{\partial}_k f(t,k,M) - \lambda_k(t)\delta_k = \rho\lambda_k(t) - \boldsymbol{\partial}_t\lambda_k(t) \tag{A.13}$$

$$C(t)^{-\gamma}\boldsymbol{\partial}_M f(t,k,M) - \lambda_M(t)\delta_M = \rho\lambda_M(t) - \boldsymbol{\partial}_t\lambda_M(t) \tag{A.14}$$

Divide by $\lambda_k(t)$ and $\lambda_M(t)$ respectively, and then use (A.8), (A.9), (A.11) and (A.12)

$$\boldsymbol{\partial}_k f(t,k,M) = \rho + \delta_k + \gamma\boldsymbol{\partial}_t \log C(t) \tag{A.15}$$

$$z_M(t)\boldsymbol{\partial}_M f(t,k,M) = \rho + \delta_M + \boldsymbol{\partial}_t \log z_M(t) + \gamma\boldsymbol{\partial}_t \log C(t) \tag{A.16}$$

Combine (A.15) and (A.16) to get an expression between k and M—which is static if $z_M(t)$ is constant

$$z_M(t)\boldsymbol{\partial}_M f(t,k,M) - \boldsymbol{\partial}_k f(t,k,M) = \delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) \tag{A.17}$$

Assume monotonicity properties such as complementarity, $\boldsymbol{\partial}_{kM} f(t,k,M) > 0$, so there is a unique M that solves (A.17) for any given k and t. Define this function as $\zeta(\cdot)$ such that,

$$M(t) \equiv z_M(t)\zeta(t,k(t)) \tag{A.18}$$

Take the derivative with the chain rule,

$$\frac{\boldsymbol{\partial}_t M(t)}{z_M(t)} = \boldsymbol{\partial}_t \zeta(t,k(t)) + \boldsymbol{\partial}_t k(t) \boldsymbol{\partial}_k \zeta(t,k(t)) + \zeta(t,k(t)) \boldsymbol{\partial}_t \log z_M(t) \tag{A.19}$$

With the law of motion for $M(t)$ in Main (38), use (A.18) and (A.19)

$$i_M(t) = \frac{\boldsymbol{\partial}_t M(t)}{z_M(t)} + \frac{\delta_M M(t)}{z_M(t)} \tag{A.20}$$

$$= \boldsymbol{\partial}_t \zeta(t,k(t)) + \boldsymbol{\partial}_t k(t) \boldsymbol{\partial}_k \zeta(t,k(t)) + \zeta(t,k(t)) \boldsymbol{\partial}_t \log z_M(t) + \delta_M \zeta(t,k(t)) \tag{A.21}$$

From laws of motion for $k(t)$ in Main (37)

$$i_k(t) = \boldsymbol{\partial}_t k(t) + \delta_k k(t) \tag{A.22}$$

Use Main (40) and (A.21) and (A.22) to solve for $\boldsymbol{\partial}_t k(t)$,

$$\boldsymbol{\partial}_t k(t) = \frac{1}{1 + \boldsymbol{\partial}_k \zeta(t,k(t))} \left(f(t,k(t),M(t)) - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t)) \zeta(t,k(t)) - \boldsymbol{\partial}_t \zeta(t,k(t)) \right) \tag{A.23}$$

From $f(\cdot)$ in (A.7) and (A.18), note that

$$f(t,k(t),M(t)) = z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{A.24}$$

Substitute into the derivatives with (A.18),

$$\boldsymbol{\partial}_k f(t,k(t),M(t)) = \frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} M(t)^{\frac{1}{\kappa-1}} \tag{A.25}$$

$$= \frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{A.26}$$

$$\boldsymbol{\partial}_M f(t,k(t),M(t)) = \frac{1}{(\kappa-1)M(t)} z(t)b(t)k(t)^{\alpha} M(t)^{\frac{1}{\kappa-1}} \tag{A.27}$$

$$= \frac{1}{(\kappa-1) z_M(t) \zeta(t,k(t))} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} \tag{A.28}$$

From (A.17)

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z_M(t) \boldsymbol{\partial}_M f(t,k,M) - \boldsymbol{\partial}_k f(t,k,M) \tag{A.29}$$

Take (A.17), (A.18), (A.25) and (A.27), substitute for the production function $f(\cdot)$, and define $\hat{M} \equiv M(t)/z_M(t)$ to get the implicit equation,

$$\delta_M - \delta_k + \boldsymbol{\partial}_t \log z_M(t) = z(t)b(t) z_M(t)^{\frac{1}{\kappa-1}} k(t)^{\alpha} \hat{M}^{\frac{1}{\kappa-1}} \left(\tfrac{1}{\kappa-1} \hat{M}^{-1} - \alpha k^{-1} \right) \tag{A.30}$$

Where the solution to this implicit equation defines the $\hat{M} \equiv M(t)/z_M(t) = \zeta(t,k(t))$ function. Substitute (A.26) into (A.15)

$$\boldsymbol{\partial}_t \log C(t) = \frac{1}{\gamma} \left(\frac{\alpha}{k(t)} z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t,k(t))^{\frac{1}{\kappa-1}} - \rho - \delta_k \right) \tag{A.31}$$

Repeat (A.23) to finalize the set of 2 ODEs in $k(t)$ and $C(t)$ given the $\zeta(\cdot)$ function from above.

$$\boldsymbol{\partial}_t k(t) = \frac{z(t)b(t)k(t)^{\alpha} z_M(t)^{\frac{1}{\kappa-1}} \zeta(t, M(t))^{\frac{1}{\kappa-1}} - C(t) - \delta_k k(t) - (\delta_M + \boldsymbol{\partial}_t \log z_M(t))\zeta(t, k(t)) - \boldsymbol{\partial}_t \zeta(t, k(t))}{1 + \boldsymbol{\partial}_k \zeta(t, k(t))} \tag{A.32}$$

The solution is then an ODE in $C(t)$ and $k(t)$ given by (A.31) and (A.32) with $\zeta(\cdot)$ defined by (A.30). The evolution of the distribution comes directly from Main (33), which in turn determines the evolution of $b(t)$. Use Main (33) and substitute with $M(t) = \zeta(t, k(t))z_M(t)$

$$\boldsymbol{\partial}_t \Phi(t, a) = -\boldsymbol{\partial}_a \Phi(t, a) + (1 - \Phi(t, a)) \left(\boldsymbol{\partial}_t \log z_M(t) + \frac{\boldsymbol{\partial}_t k(t) \boldsymbol{\partial}_k \zeta(t, k(t)) + \boldsymbol{\partial}_t \zeta(t, k(t))}{\zeta(t, k(t))} + \delta_M \right) \tag{A.33}$$

□

In Proposition 1, the equilibrium choices have been simplified to a system of 2 ODEs in $k(t)$ and $C(t)$, where given a particular $M(t)$ path, the corresponding $\Phi(t, a)$ evolution. Keeping track of $\Phi(t, a)$ is crucial since net-entry and expansion of $M(t)$ leads to changes in the age distribution, especially during transitions.

A.2 Stationary Equilibrium

Proof of Main Proposition 6. From Main Appendix B.3, the stationary age distribution is independent of the choice variables k and M. As derivatives of the $\zeta(\cdot)$ function drop out in the stationary equilibrium, it can be written in terms of M rather than $\hat{M}$ and $\zeta(\cdot)$. From Proposition 1, the stationary set of equations is,

$$\delta_M - \delta_k = zbk^{\alpha} M^{\frac{1}{\kappa-1}} \left(\frac{z_M}{(\kappa - 1)M} - \frac{\alpha}{k} \right) \tag{A.34}$$

From this $\hat{M} = \zeta(k)$ function, the equilibrium capital solves

$$\rho + \delta_k = \alpha z b k^{\alpha-1} M^{\frac{1}{\kappa-1}} \tag{A.35}$$

And given the k and M, from $i_k = \delta_k k$ and $i_M = \delta_M M/z_M$, the equilibrium C is

$$C = zbk^{\alpha} M^{\frac{1}{\kappa-1}} - \delta_k k - \delta_M M/z_M \tag{A.36}$$

□

Appendix B Awareness Evolution Examples

To give two additional examples of awareness processes (nested by Main Example 1).

Example 1 (Baseline Awareness Model). *Assume that each consumer has an intensity $\theta > 0$ of becoming aware of a firm in an industry, and an equal probability of becoming aware of a particular firm (including the potential of repeating a meeting with an existing firm in her information set, which doesn't add to the count). Also assume that customers can forget an existing firm at rate $\mu \geq 0$. Then, the infinitesimal generator is*

$$\mathbb{Q} = \begin{bmatrix} -\theta & \theta & 0 & \dots & & \dots & 0 \\ \mu & -\left(\frac{N-1}{N}\theta + \mu\right) & \frac{N-1}{N}\theta & 0 & \dots & \dots & 0 \\ 0 & \mu & -\left(\frac{N-2}{N}\theta + \mu\right) & \frac{N-2}{N}\theta & 0 \ \dots & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots \ \mu & -\left(\frac{1}{N}\theta + \mu\right) & \frac{1}{N}\theta \\ 0 & 0 & 0 & 0 & \dots \ 0 & \mu & -\mu \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)} \tag{B.1}$$

Example 2 (Baseline Large N Approximation)**.** *Take the limit as $N \to \infty$ of Example 1. Then, the generator is*

$$\mathbb{Q} = \begin{bmatrix} -\theta & \theta & 0 & & \dots \\ \mu & -(\mu+\theta) & \theta & 0 \dots & \\ 0 & \mu & -(\mu+\theta) & \theta\ 0 & \dots \\ \dots & & & & \dots \end{bmatrix} \tag{B.2}$$

In queueing theory, this is the classic M/M/1 queue, or a simple Poisson birth-death process (see Kleinrock (1975) section 3.2).[1] *More generally, define $\omega \equiv \theta/\mu$, $\alpha \equiv 2\sqrt{\theta\mu}$, and $I_n(\cdot)$ as the modified Bessel function of the first kind. Transition dynamics from Kleinrock (1975) page 77 are*

$$f_n(a) = e^{-(\theta+\mu)a}\left[\omega^{\frac{n}{2}} I_n(\alpha a) + \omega^{\frac{n-1}{2}} I_{n+1}(\alpha a) + (1-\omega)\omega^n \sum_{j=n+2}^{\infty} \omega^{-j/2} I_j(\alpha a)\right]$$

From Abate and Whitt (1987), A first-order approximation for the first moment during transition is

$$\mathbb{E}_a[n] \approx 1/\omega - \alpha_3(\omega) e^{-a/\alpha_2(\omega)} \tag{B.3}$$

where $\alpha_2(\omega) \equiv 2(1-\omega)^{-2}\alpha_4(\omega), \alpha_3(\omega) \equiv \omega(1-\omega)^{-1}(1+2\alpha_4(\omega))^{-1}$, and $\alpha_4(\omega) \equiv \frac{2+\omega+(5-(1-\omega)(5+\omega))^{1/2}}{4}$,

For the case where $\mu = 0$, this is a Poisson counting process with no stationary distribution. In transition, $\mathbb{E}_a[n] = \theta a$, and $f_n(a) = \frac{e^{-\theta a}(\theta a)^n}{n!}$.

Appendix C Awareness with Fully Differentiated Firms

This section provides some details on solving the model with differentiated firms, include heterogeneity in quality and entry-time.

C.1 Cardinality and Notation

Ignoring any symmetry on entry timing or intrinsic quality between firms, the possible states for awareness A are $\mathbf{2}^{\mathcal{I}}$, with cardinality

$$\mathbf{N} \equiv \left|\mathbf{2}^{\mathcal{I}}\right| = \sum_{k=0}^{N} \binom{N}{k} \tag{C.1}$$

Then the awareness state in an industry of age a is $\hat{f}(a, A) : \mathbb{R}^{\mathbf{N}} \to \mathbb{R}$, with $\sum_{A \in \mathbf{2}^{\mathcal{I}}} \hat{f}(a, A) = 1$. To create a Markov chain, I need to establish a consistent ordering of the states in $\hat{f}(a, A)$. Define the lexicographic ordering matrix as

$$\mathbf{L} \equiv \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix}_{\mathbf{N}\times N} \tag{C.2}$$

In (C.2), each row represents a possible awareness state for an individual, with the N firms ordered on the horizontal axis. For a given row, a 0 denotes that the firm is not in that awareness state, and 1 denotes that it is. Table 1 provides an example of the mapping. Additionally, define the following vectors and matrices:

[1] The stationary distribution of awareness set sizes is geometric and non-degenerate if $\theta < \mu$, with $\mathbb{E}_\infty[n] = \theta/\mu$ and $f_n(\infty) = (1-\theta/\mu)(\theta/\mu)^n$. See Kleinrock (1975) page 77 for transition dynamics.

1. $\mathbf{I}$ is the diagonal matrix of size $N \times N$
2. $\vec{\mathbf{1}}$ is the vector of 1's of size N
3. $\mathbf{e}_\iota$ is a vector of 0's with a 1 at the ι'th position

ι Index	Set $A_j(t)$	Row of L
1	$\{\emptyset\}$	$\begin{bmatrix}0 & 0 & 0 & 0 & \dots & 0\end{bmatrix}$
2	$\{1\}$	$\begin{bmatrix}1 & 0 & 0 & 0 & \dots & 0\end{bmatrix}$
3	$\{2\}$	$\begin{bmatrix}0 & 1 & 0 & 0 & \dots & 0\end{bmatrix}$
...		
$N+2$	$\{1,2\}$	$\begin{bmatrix}1 & 1 & 0 & 0 & \dots & 0\end{bmatrix}$
$N+3$	$\{1,3\}$	$\begin{bmatrix}1 & 0 & 1 & 0 & \dots & 0\end{bmatrix}$
...		
$\mathbf{N}$	$\{1,2\dots N\}$	$\begin{bmatrix}1 & 1 & 1 & 1 & \dots & 1\end{bmatrix}$

Table 1: Lexicographic Ordering of States

C.2 Example Stochastic Process

Example 3 in this section will modify the process in Main Example 1 to allow for full differentiation of firms in the $\hat{f}(a, A)$ evolution process. The infinitesimal generator of the Markov chain (no longer just a count process) is denoted $\hat{\mathbb{Q}}$, and for simplicity I assume there is no forgetting (i.e., $\mu = 0$).

Example 3 (Baseline Awareness Model With Full Differentiation)**.** *As in Main Example 1, assume each consumer has a $\theta > 0$ probability of becoming aware of some firm in an industry, and an equal probability of becoming aware of a particular firm (including the potential of repeating a meeting with an existing firm in there information set, which doesn't change awareness) and word of mouth diffusion. The infinitesimal generator is $\hat{\mathbb{Q}}(a) \in \mathbb{R}^{\mathbf{N}\times\mathbf{N}}$, where the (ι, ι') element is of the Markov process is given by*

$$\hat{\mathbb{Q}}_{\iota\iota'}(a) = \begin{cases} -(\theta + \theta_d(1 - \hat{f}_0(a)) & \iota' = \iota = 1 \\ \frac{\theta+\theta_d(1-\hat{f}_0(a))}{N} & \iota = 1, 1 < \iota \leq N \\ -\theta + \frac{\theta}{N}\mathbf{e}_\iota \mathbf{L}\,\vec{\mathbf{1}}^T & \iota' = \iota > 1 \\ \frac{\theta}{N} & (\mathbf{e}_{\iota'} - \mathbf{e}_\iota)\mathbf{L} = 1 \\ 0 & o.w. \end{cases} \tag{C.3}$$

Awareness evolves through the solution to the following system of differential equations given the initial condition $\hat{f}(0)$,

$$\boldsymbol{\partial}_a \hat{f}(a) = \hat{f}(a) \cdot \hat{\mathbb{Q}}(a) \tag{C.4}$$

In the case with no word-of-mouth diffusion, $\theta_d = 0$, (C.3) simplifies to the age-invariant Markov-chain

$$\hat{\mathbb{Q}}_{\iota\iota'} = \begin{cases} -\theta + \frac{\theta}{N}\mathbf{e}_\iota \mathbf{L}\,\vec{\mathbf{1}}^T & \iota' = \iota \\ \frac{\theta}{N} & (\mathbf{e}_{\iota'} - \mathbf{e}_\iota)\mathbf{L} = 1 \\ 0 & o.w. \end{cases} \tag{C.5}$$

and the unconditional distribution of awareness in the economy with $\theta_d = 0$ in (C.4) evolves from the initial condition as,

$$\hat{f}(a) = \hat{f}(0) \cdot e^{a\hat{\mathbb{Q}}} \in \mathbb{R}^{\mathbf{N}} \tag{C.6}$$

As previously discussed, if there is any symmetry in the set of firms, this process can be simplified substantially. Regardless, systems of ODEs of the form (C.4) can be solved numerically for *very* large $\mathbf{N}$.

C.3 Mapping Duopoly Example to the Count Distribution

This section takes a simple symmetric duopoly with firms $\{1, 2\}$ and connects Example 3 with $\hat{f}(\cdot)$ and $\hat{\mathbb{Q}}$, to Main Example 1 with $f(\cdot)$ and $\mathbb{Q}$, look at a duopoly.

From Example 3, the pmf is $\hat{f}(a, A)$ for A accorded according to $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and with the infinitesimal generator,

$$\hat{\mathbb{Q}}(a) \equiv \begin{bmatrix} -(\theta + \theta_d(1 - \hat{f}_0(a))) & (\theta + \theta_d(1 - \hat{f}_0(a)))/2 & (\theta + \theta_d(1 - \hat{f}_0(a)))/2 & 0 \\ 0 & -\theta/2 & 0 & \theta/2 \\ 0 & 0 & -\theta/2 & \theta/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From these, define the count distribution as: $f_0(a) \equiv \hat{f}(a, \emptyset)$, $f_1(a) \equiv \hat{f}(a, \{1\}) + \hat{f}(a, \{2\})$, and $f_2(a) \equiv \hat{f}(a, \{1, 2\})$. The generator becomes identical to Main Example 1 with $N = 2$,

$$\mathbb{Q}(a) \equiv \begin{bmatrix} -(\theta + \theta_d(1 - f_0(a))) & \theta + \theta_d(1 - f_0(a)) & 0 \\ 0 & -\theta/2 & \theta/2 \\ 0 & 0 & 0 \end{bmatrix}$$

C.4 Customers and Market Share

Proposition 2 (Number of Customers for Gumbel Preferences). *With $\xi_j \sim$ Gumbel:*

$$x_i(a, p) = \sum_{A\,|\,i} \left[\hat{f}(a, A) \frac{\left(\frac{p_i}{q_i}\right)^{-1/\sigma}}{\sum_{i' \in A} \left(\frac{p_{i'}}{q_{i'}}\right)^{-1/\sigma}} \right] \tag{C.7}$$

Proof of Proposition 2. The proof follows standard derivation of market shares in discrete choice theory. Define the number of customers as the measure of consumers choosing product i from Main (A.3):

$$x_i(a, p) = \int \mathbb{1}\{\log\left(\tfrac{p_{i'}}{q_{i'}}\right) - \log(\tfrac{p_i}{q_i}) > \sigma(\xi_{i'j} - \xi_{ij}) | \forall\, i' \in A_j \setminus i\} \, \mathrm{d}\hat{\Psi}(A_j, \xi_{1j}, \xi_{2j}, \ldots \xi_{Nj}) \tag{C.8}$$

Using the independence of A and ξ_j, the integral can be split using Fubini's Theorem over all awareness states that contain product i, i.e., $\{A\,|A \in \mathbf{2}^{\mathcal{I}} \text{ s.t. } i \in A\}$

$$x_i(a, p) = \sum_{A\,|\,i} \hat{f}(a, A) \int \mathbb{1}\{\log\left(\tfrac{p_{i'}}{q_{i'}}\right) - \log(\tfrac{p_i}{q_i}) > \sigma(\xi_{i'j} - \xi_{ij}) | \forall\, i' \in A \setminus i\} \mathrm{d}G(\xi_j) \tag{C.9}$$

This this can be reorganized as

$$x_i(a, p) = \sum_{A\,|\,i} \hat{f}(a, A) \int \mathbb{1}\{-\tfrac{1}{\sigma} \log(\tfrac{p_i}{q_i}) - \tfrac{1}{\sigma}(-\log\left(\tfrac{p_{i'}}{q_{i'}}\right)) > \xi_{i'j} - \xi_{ij} | \forall\, i' \in A \setminus i\} \mathrm{d}G(\xi_j) \tag{C.10}$$

This form is of the same structure as many discrete choice problems. Given a particular functional form of $g(\xi_j)$, this can be solved numerically or analytically. Under the assumption that $g(\xi_j)$ is iid Gumbel, this can be solved in closed form, following Anderson, De Palma, and Thisse (1992).

$$x_i(a,p) = \sum_{A\,|\,i} \left[\hat{f}(a,A) \frac{\left(\frac{p_i}{q_i}\right)^{-1/\sigma}}{\sum_{i' \in A} \left(\frac{p'_i}{q_{i'}}\right)^{-1/\sigma}} \right] \tag{C.11}$$

□

To compare to standard discrete choice results, this is a weighted sum of the market shares for every type of awareness set. Note that σ has no effect on market share if p_i are identical for all firms. As the choices are deterministic and conditional on a particular information set, the aggregated market shares cannot be easily interpreted as choice probabilities, as is typical in discrete choice.

Appendix D Additional Aggregate Empirics

Continuing with the evidence in Main Section 2.1, I document more patterns of trademark and patent obsolescence in Figure 1, and evidence on the role of sales, marketing, and advertising in Figure 2. See Appendices F.1 and F.2 for details on the life cycle of a trademark and patent.

D.1 Patent and Trademark Indicators

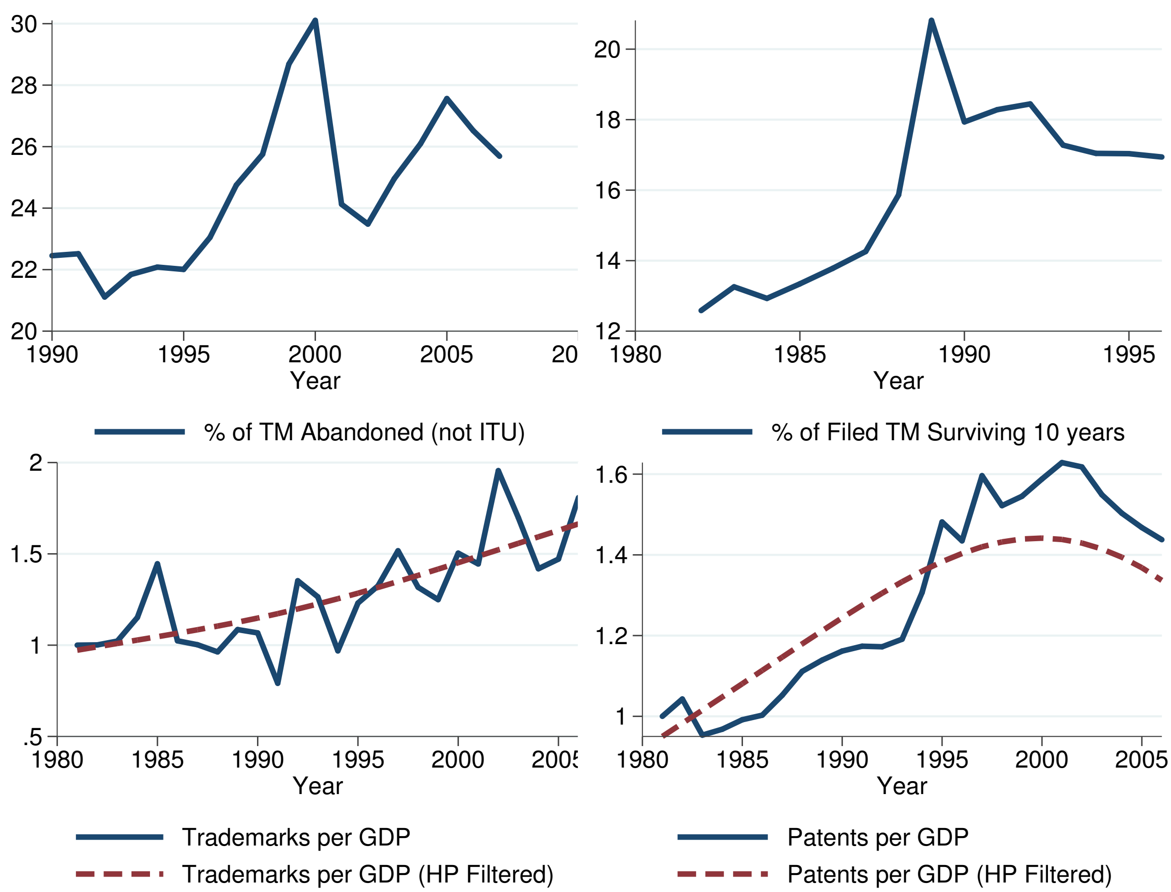

Figure 1: More on Trademarks and Patents

In order to get a sense of whether the new ITU trademark type breaks the evidence of increasing abandonment in Main Figure 2, I show the abandonment rates for non-ITU trademarks in Figure 1. The data shows increasing abandonment even without the ITU trademarks (although part of the jump in Main Figure 2 around 1990 is likely due to this administrative change, it is tough to separate it out from the 1990 recession). Curiously, combined with the increasing abandonment rates, Figure 1 also shows that *conditional on not being abandoned*, trademarks seem to survive longer—moving somewhere between 13% and 17% during this period. The interpretation of trademarks as product identifiers suggests that a more complicated model with a non-constant hazard from an age dependent δ_M—likely driven by some sort of selection—would drive other interesting secular changes.

In terms of the role of trademarks and patents in the economy, Figure 1 shows that, from the 1980s to the early 2000s, the ratio of trademarks registrations per GDP raises by more than 50%, and the ratio of patents per GDP rose by almost 40%. At first, this might suggest that due to diminishing returns to scale in R&D, the amount of research required to generate a new product has been increasing, but keep in mind that the number of trademarks required has actually increased by more—and trademark applications are a better proxy than patents for the number of *new* varieties and/or products. So, either the number of products required to generate a unit of GDP is growing, or a large number of products are constantly becoming obsolete. The evidence in Figure 1 and Main Figure 2 suggests that obsolescence it a key part of the story—and that it has been increasing over time.

D.2 Sales, Marketing, and Advertising Indicators

When considering the role of endogenous expansion of awareness in Main Section 7, it is important to look at aggregate patterns of investment to determine the likely comparative statics that could lead to secular changes. For this, I show several proxies: (1) total Compustat advertising to revenue ratio (XAD / REVT); (2) total Compustat sales, general, and administrative to revenue ratio (XSGA / REVT); and (3) total media spending to GDP ratio in the US from WARC.

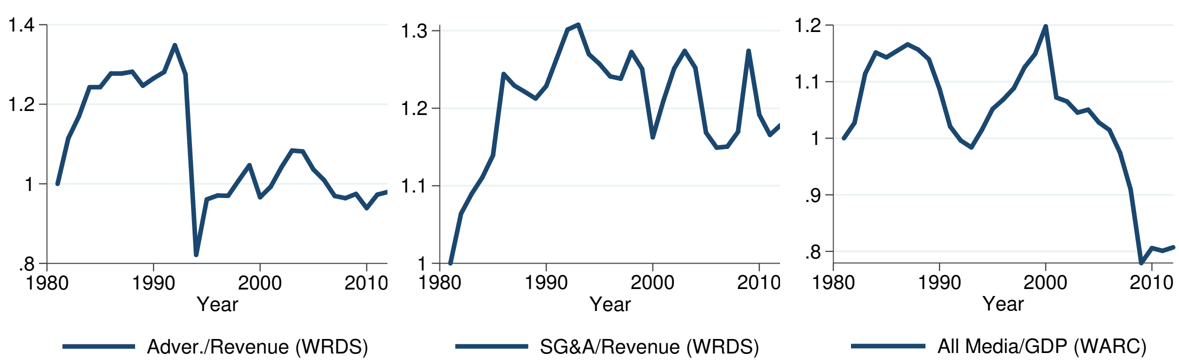

Figure 2: Advertising, SG&A, and Media Expenditures (from Compustat and WARC)

None of these is perfect. The evidence from Compustat XAD is the most suspicious since firms have no obligation to report or accurately categorize advertising spending in their audited financials (which is why there are so many 0's in the XAD data). On the other hand, the SG&A reporting in compustat includes all XAD, as well as any other non-operating expenses, and is a reporting requirement in the audited financials. In that sense, it is a better proxy than XAD because it does not have the same selection issues, but it does conflate in other overhead expenses in addition to R&D (although reported R&D can be removed with XRD and is typically a small fraction of revenue). The WARC data provides a more complete snapshot without the selection bias of Compustat firms, but applies only to advertising. I suspect that advertising is only a small portion of the aggregate spending on sales and marketing activities outside of consumer product firms—as it does not include things like salesperson commissions and salaries, travel expenses to meet customers, production of marketing materials, customer loyalty programs, etc.

Regardless, the patterns of these proxies are inconclusive, and provide some evidence for changes in the sales and marketing productivity, ν, in Main Section 7.2, and some evidence for the role of a changing obsolescence rate δ_M. But as the sales and marketing to GDP ratio in both comparative statics changes only modestly, the data in Main Figure 3 makes a δ_M change more consistent with the evidence on changes in factor shares.

Appendix E Robustness

This section examines robustness in the panel data regressions.

E.1 Robustness to Controls

If controls are not used, then manufacturing industry data is not required and the number of available industries increases from 189 to 502. While the lack concentration data should bias markups to increase with age (since concentration tends to increase with industry age), the effect does not seem strong. The key results are in Figure 3.

Figure 3: Effects of Age Relative to Peak Employment (Only Year Fixed Effects)

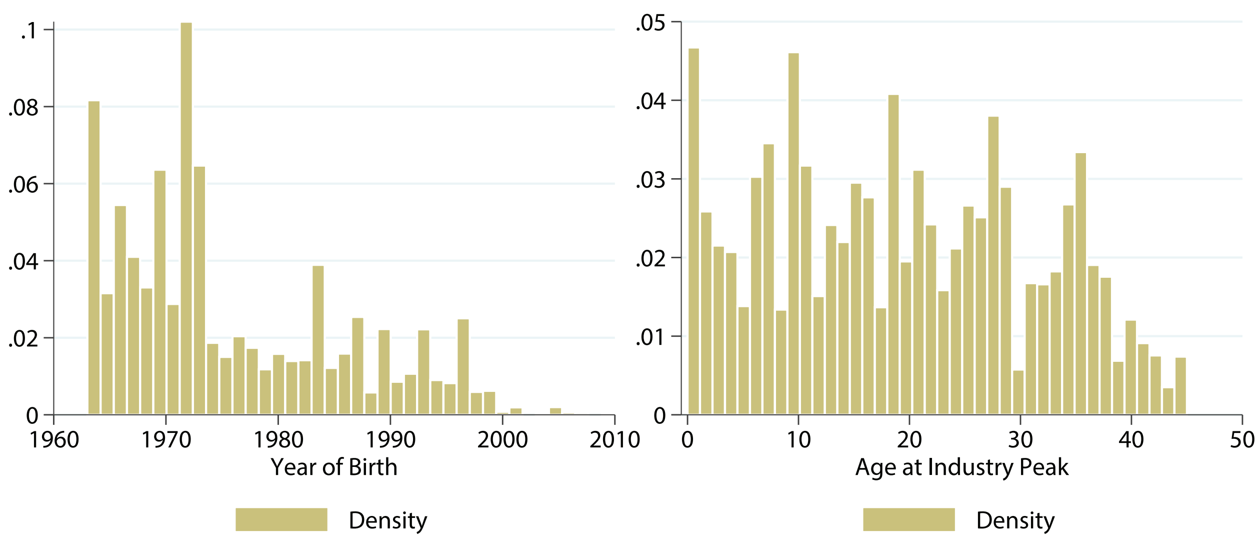

Figure 4: Histogram of Birth Year and Peak Employment Year within Sample (Only Year Fixed Effects)

E.2 Robustness to Alternative Birth Definition

As the baseline uses employment as the proxy for industry birth (and peak). The result is generally robust to alternative definitions. For example, using the relative enterprise value to the total enterprise value in sample, gives Figure 5

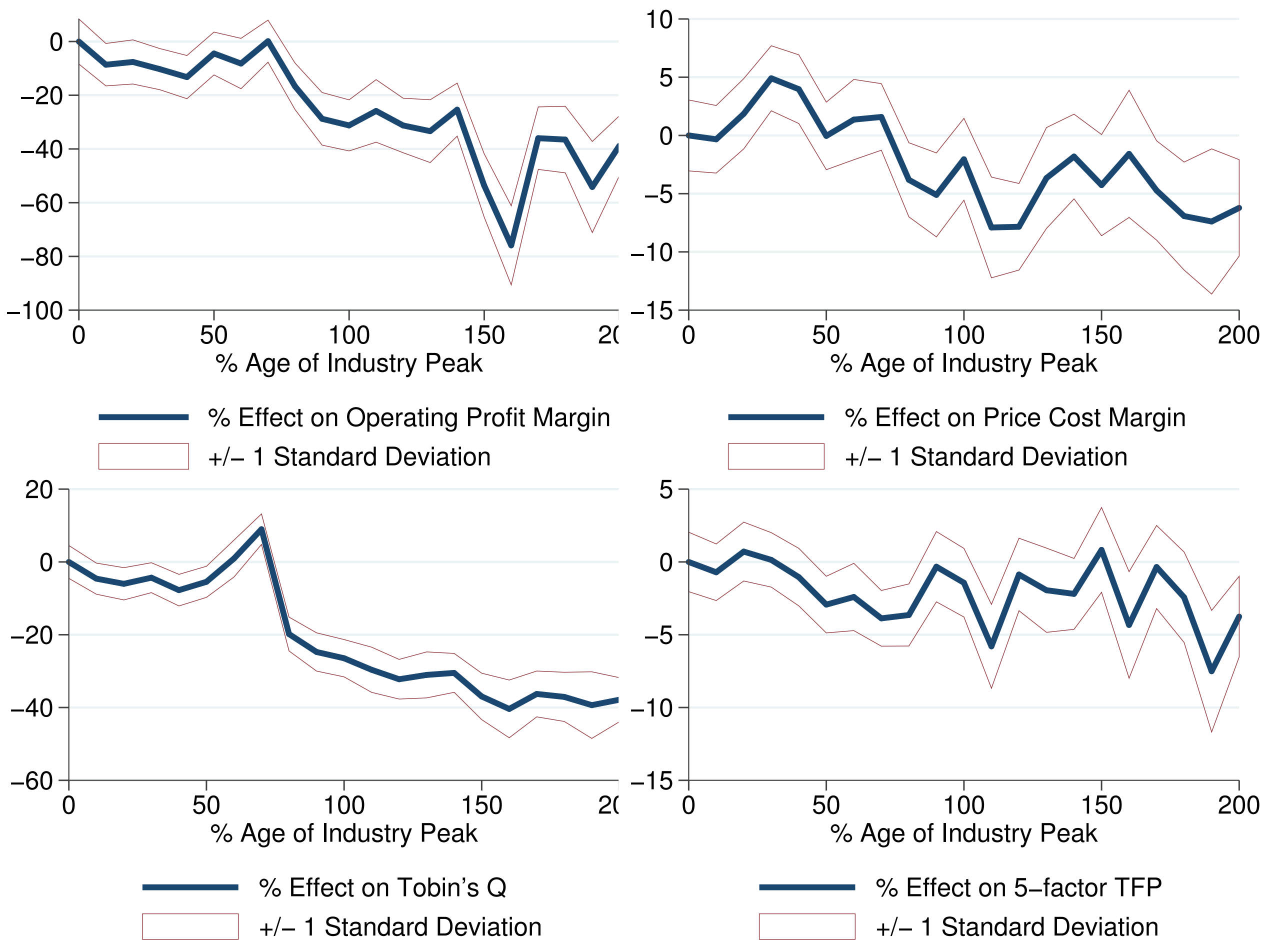

Figure 5: Effects of Age Relative to Peak Relative Enterprise Value

Appendix F Data

The following provides more details on how the data is constructed for the empirical results of Main Section 2. See Main Appendix F for the summary of all data sources, and where they are used.

F.1 Trademark Life Cycle

The primary reference for the trademark data is Graham, Hancock, Marco, and Myers (2013).[2] A trademark can go through the following events:

[2]Thanks to Amanda Myers at the USPTO for help with the data and interpretation.

- **Filed**: A trademark (typically used to identify a product or service) is filed with the USPTO. After 1989, they can be filed with an "Intent to Use" (ITU), which is a placeholder for a trademark that one has not used, but intends to within six months.

- **Abandoned**: The trademark may be abandoned *prior to registration*. This could occur because registration fees are not paid (very common) or because the office rejects the trademark (unlikely).

- **Registered**: The trademark is registered by the trademark office, and is now in force

- **Maintained**: After six years of the trademark being registered, the owner needs to demonstrate that it is being used in order to maintain it.

- **Canceled**: A registered trademark ceases to exist, and is no longer in force. This could occur between registration and maintenance, or between renewals.

- **Renewal**: Unlike patents or copyrights, trademarks can be renewed indefinitely (as long as fees are maintained). For trademarks up for renewal prior to 1989, the renewals are every 20 years. For renewal events after 1990, they are every ten years.

While a cancellation could happen due to failure to prove usage in the maintenance event, this is rare and happens primarily because the owner decides that the fees are not worth it (perhaps because revenue from the product is insufficient, or the product ceases to exist). The change in the administrative procedures also shows that we need to be careful when comparing pre- and post-1989 data due to the change to ten-year renewals, and the new ITU type of trademark. However, there do not seem to be large secular changes in administrative procedures that would change the incentives for having trademarks—other than the value of the products themselves.

F.2 Patent Life Cycle

The primary references for the two sources of patent data are Marco, Carley, Jackson, and Myers (2015) and Graham, Marco, and Miller (2016). A patent can go through the following events:

- **Filed:** The invention has been filed with the USPTO, but has not completed any examination.

- **Abandoned:** The patent is abandoned prior to being issued. This could happen because the filer stop paying fees, decide not to attempt modifications for a non-final rejection, etc.

- **Issued:** The patent is issued and is in force.

- **Expired (Due to Non-Payment):** The owner of an issued patent stopped paying maintenance fees, and the patent expired prior to the end of its term.

- **Expired (End of Term):** Patents are valid only for 20 years from the filing date (or a more complicated calculation prior to 1995, irrelevant for our purposes).

- **Rejected, Pending, etc.:** Most patent applications are initially rejected and need to be revised. Final rejections are more rare, and patents often will be abandoned prior to that point. A variety of other events can happen, but the above lists the most important for our analysis.

The abandonment rate is useful as a proxy for rapid obsolescence since the products using a patent may cease to be useful prior to the completion of the examination process. The issue is that it conflates patents likely to be rejected (abandoned prior to a final rejection) with patents likely to be issued (but still not sufficiently valuable). Patents that expire due to non-payment are another proxy. These are issued patents for which the owner decided not to keep up the maintenance fees, required during the 20-year term of the patent.[3]

For interpreting the time series in Main Figure 2, a few dates are necessary to keep in mind: (1) in 1995, inventors were allowed to file provisional patent applications, which were intended to provide a way to get early filing dates without writing down any specific claims or initiating the examination process; (2) in 1995, the term of the patents was extended from something between 17-20 years to a uniform 20 years. While this may have had some effect on the incentives to patent, I suspect that they are 2nd order; and (3) in the early 1990s, the USPTO began to allow "methods of doing business" as patents, which may have increased the types of processes that could be patented. By the mid-2000s, this policy began to be reversed. Finally, as documented in Akcigit, Celik, and Greenwood (2016), the financial incentives for having patents as an option value may have changed due to secondary financial markets.

For all of these reasons, I am suspicious of comparing different years in the time series of patent data as a proxy for the creation/obsolescence of products and, instead, prefer trademarks—which are almost always granted and more directly represent a product. Nevertheless, the pattern of increasing obsolescence with patent data in Main Figure 2 is very strong.

F.3 Aggregate Data

The aggregate data uses a combination of IP data from the USPTO and more standard national accounts data from FRED.

USPTO Trademark Data The trademark data provides a great deal of information on individual case files:

- Download from `http://www.uspto.gov/learning-and-resources/electronic-data-products/trademark-case-files-dataset-0`
- The CFH status codes descript the current state of the trademark. See table 1 on page 47 of `http://www.uspto.gov/sites/default/files/products/tmdailyapp-documentation.pdf`
- The USPTO went through a process of computerization in the early 1980s. While individual case files are often backfiled before this, the data on abandonment and cancellation is insufficient. Hence, remove all data prior to 1981.
- Further clean the data by removing the CFH codes 622, 636, or 618 (which represent backfiled data, insufficient for our purposes), and if the filing date is missing.
- Otherwise, `abandon_dt` field provides date of abandonment (if applicable), the `reg_cancel_dt` date field provides a cancellation date (if applicable), and `expired_dt` provides the expiry date (if applicable). If the date field is missing, but the `cfh_status_cd` is a code for abandoned, canceled, or expired, then the `cfh_status_dt` can be used as a proxy for the date of that event. See Appendix F.1 for detailed on the terminology and lifecycle.
- Using the above date fields, the `filing_dt`, and the `registration_dt`, I can track proportions of trademarks filed in a particular year which are abandoned, canceled, etc.

[3]See `https://www.uspto.gov/learning-and-resources/fees-and-payment/uspto-fee-schedule` for the fee schedule.

USPTO Patent Data The patent data provides details on individual patent applications:

- See Appendix F.2 for a description of the life cycle of a patent and terminology.
- For the most part, the public-pair data is only required to analyze more detailed robustness checks on lengths of time for different events, etc. The hisorical dataset:
  - Download from `https://data.uspto.gov/data2/patent/historical/2014/`
  - For our purposes, the `historical_masterfile.dta` file is sufficient.
  - The `disp_ty` provides information on whether a patent is abandoned (`ABN`), issued (`ISS`) or pending (`PEN`)
  - As in the trademark data, drop any data pre-computerization before 1981 since backfiled data is insufficient for calculating survival. Also, since the proportion of patents pending is too high for more recent data, drop any patents filed after 2009.
  - Given the application date, `appl_dt`, the `disp_ty` code is sufficient to calculate the basic figures in the document.
- The public-pair database allows us to track more details, such as expiry for non-payment and the timing of events.
  - Download from `http://www.uspto.gov/learning-and-resources/electronic-data-products/patent-examination-research-dataset-public-pair`
  - The data file `application_data.dta` is sufficient for my needs, though the `status_codes.dta` is useful to see the description of events.
  - I focus on regular applications by ensuring only `REGULAR` values in the `application_type`.
  - The `disposal_type` provides high level data on the eventual status of the patent. For example, `disposal_type` = `ISS` means the patent was issued, and `patent_issue_date` provides when it occurred. `appl_status_code` provides more details on the relevant events.
  - Another useful flag is if a patent was issued, i.e. `disposal_type` = `ISS` but expired due to nonpayment, i.e. `appl_status_code` = 250
  - If the `disposal_type`= `ABN`, then the patent application was abandoned. Sometimes this field is missing, in which case the `appl_status_code` provides abandoned status codes.

WARC Data on Advertising In the appendix, further analysis is done with advertising data.

- Download the US Ad Spending from `http://www.warc.com/Pages/ForecastsAndData/InternationalData.aspx?Forecast=DatabaseAndCustomTables&isUSD=True`
- The data begins in 1980. For our purposes, All Media spending is appropriate.

FRED Standard data sources are downloaded from `https://research.stlouisfed.org`. A few of the sources worth mentioning:

- Tobin's Q is calculated as: MVEONWMVBSNNCB / TNWMVBSNNCB
- Corporate Profit Share is: W273RE1A156NBEA
- Labor shares are: PRS84006173, PRS85006173, PRS88003173
- Market Capitalization is: NCBEILQ027S

F.4 Panel Data

The panel data primarily uses the combination of the Census MID, Census Concentration Ratios, and Compustat.

Census Concentration Ratios The concentration ratios are prepared by the US Census.

- Most of the data comes from `https://www.census.gov/econ/concentration.html`. In particular, the 1947-1992 SIC4 data comes from the downloaded spreadsheet on that site
- The 1997 data comes from the PDF on that site. The data was digitized, as it only seemed to be in the document form (please email me for the data)
- The 2007 and 2002 can be downloaded directly as well from `https://www.census.gov/econ/concentration.html`, (weighted by value added as well as value of shipments)
- The 2012 data comes from: `http://factfinder.census.gov/faces/nav/jsf/pages/searchresults.xhtml?refresh=t`

In order to consolidate all of the concentration data for the panel, concordance tables are used.

- This requires SIC4 to NAICS1997 conversion, as well as the newer NAICS codes to NAICS1997 for consistency. All of the NAICS concordance files are from: `http://www.census.gov/eos/www/naics/concordances/concordances.html`
- When converting between the SIC in 1947 to 1972 data and NAICS, I can use the weighting method in the NBER-Census MID database. See `http://www.nber.org/nberces/nberces5809/conc_sic87_naics97_documentation.pdf` for a description.
- The files are available online from `http://www.nber.org/data/nberces5809.html`, but Wayne Gray and Randy Becker manage a cleaned up version of the data.

For the data used in my regressions,

- More recent concentration data is weighted by both the value of shipments, and the value-added. As the historical data is only weighted by shipments, I need to stick with that method.
- The number of firms in the industry is the `comp` variable.
- I checked several measures of concentration as tests, but the share of the value of shipments in the top 8 firms, `con8` provides the baseline measure of concentration.
- The concentration data is only sampled at 4 year intervals. As using concentration as a control would significantly decrease my number of observables, I do my baseline model with interpolation of the `comp` and `con8` to provide data for every year. As I also do the regressions without interpolation, and even without controls, I do not think this is driving any key results.

Age Bins As discussed, different industries may have different lengths of life cycles (as evident in the time to peak). For a given variable used to generate age, such as the total employment of the industry, we can compare regressions using the normalized vs. direct age. To normalized:

- Choose the variable to bin with (e.g., employment in our baseline)
- Calculate the "birth year" of each industry as when that variable it attains a certain threshold (e.g., employment reaches 5% of its eventual maximum)

- Calculate the "age at peak" as the year at which the maximum value is attained minus the birth year.
- Calculate the relative age as: (year - birth year)/(age at peak)
- As these ages are now a floating point number, we would be unable to look at marginal effects. Choose bins (e.g., deciles) and create an integer age bin for industries with ages in that range.
- Using the age bins, use dummy variables for the age bin to find the effects of age as a proportion of the maximum age.

NBER-Census Manufacturing Industry Database (MID) See Bartelsman, Becker, and Gray (2000) for detailed on the data source. The MID provides a panel from 1958, providing details on TFP, value added, value of shipments, payroll, etc.

- The files are available online from `http://www.nber.org/data/nberces5809.html`, but Wayne Gray and Randy Becker manage a cleaned up version of the data.
- I use this data for several markup calculations, such as the price-cost-margin, which is calculated according to Domowitz, Hubbard, and Petersen (1986) as described in Nekarda and Ramey (2011). The calculation is `pcm = (vship+D.invent-pay-matcost)/(vship+D.invent)`
- Another method of calculating markups is the inverse share of value-added over wages, in the same sense as Hall (1988). The calculation is: `vadd / prodw`
- Finally, TFP comes from the five-factor TFP, `tfp5`

CRSP/Compustat Compustat allows other calculations of markups and Tobin's Q from the data, aggregated up by NAICS code.

- Data is downloaded from `https://wrds-web.wharton.upenn.edu`, at an annual frequency.
- The calculations of profits, etc. closely follow Gorodnichenko and Weber (2016)
- The Price to cost margin (PCM) is calculated here as the ratio of net sales minus costs of goods sold `revt/COGS`
- The operating profit margin is calculated as the operating income before depreciation / total revenue, `oibdp / revt`
- Tobin's Q is the Market Value of Assets / Total assets, `(csho * prcc_c - ceq + at)/at`
- In all cases, panel regressions are on the collapsed values to the NAICS code, with the OPM, PCM, and Tobin's Q as the mean within that industry.

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