Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier Online Technical Appendix

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Appendix A Stochastic Process Alternatives

Innovation in this model is a controlled stochastic process for Z. The purpose of the L/H Markov chain with endogenous (but deterministic) growth rates of Z in the H state is to provide the **simplest** process to ensure the following:

1. *Stochastic.* As shown in Appendix E.5, if the process for Z is deterministic and the distribution starts off with bounded support, the only equilibrium are those with no long-run technology adoption and all firms growing at the deterministic innovation rate (even if it is endogenous).

2. *Maintain a finite frontier (i.e., maximum Z) at all t.* This is necessary since the evolution of the frontier is central to our questions, and because we find that solutions with an infinite frontier (either from the stochastic process, or from infinite initial conditions) are fundamentally different

3. *Measurable stochastic process for Z, over time.* Otherwise, we can not express the optimal decisions of agents (i.e., Bellman equations and the Feynman-Kac formula need to be defined). For example, while one can write a continuous time stochastic process that is IID, there are no such processes that are both measurable (along the time dimension) and non-deterministic.

4. *Have the growth rate determined by endogenous agent decisions at the frontier.* As we will look at endogenize growth rates, the frontier must grow by the decisions of those with the highest Z rather than through some exogenous process.

5. *Orthogonal to technology diffusion.* While we add in a combined innovation-diffusion process later in our model, we want to keep them as orthogonal as possible to isolate the interactions. In particular, if the innovation process is directly a function of the $\Phi(t, Z)$ distribution, then it cannot be orthogonal.

[1] References to equations, etc. in the main paper are prefixed by *Main Paper*. An earlier version of this paper was circulated under the title "The Growth Dynamics of Innovation, Diffusion, and the Technology Frontier."

6. *Clarity of the maximum innovation rate.* In order to analyze the role of "latent growth", we want to have a clear sense of the maximum innovation rate of agents in the economy. If possible innovation rates are unbounded, then it is more difficult to determine the elements of growth coming from initial conditions rather than directly from the innovation choices of agents.

While we don't solve a version of this model with a finite number of agents, it is a good mental test of the process. For example, consider applying a process to an economy with 3 operating firms, where the frontier is the productivity of the highest Z agent.

Processes that Fulfill the Requirements. The following are the only processes we have found that fulfill these requirements,

1. Our method of augmenting the state Z with an additional discrete state evolving through a Markov Chain. The discrete state provides a deterministic growth rate—which can be controlled. The maximum growth rate in all of the discrete states is bounded.

2. Add in an auxiliary continuous state, g, for the firm, which follows some bounded stochastic process with some persistence (e.g., Brownian motion reflected at some maximum). This is the deterministic growth rate for the firm at any point in time—i.e., the drift rate in logs. The only difficulty is in having the process controlled, since the optimal choice would need to change the reflection barrier in order to endogenize the growth rate of the frontier.

3. Have a Poisson arrival of multiplicative **backwards** jumps, e.g., δZ where $\delta < 1$, and allow the firm to choose a deterministic drift otherwise. This adds in necessary stochastic element, and the growth rate of the frontier is simply the drift chosen by the frontier. The benefit of this approach is that we do not need to augment the state, but the cost is a far more complicated ODE—and questionable economics due to the absolute loss in productivity.

The first two approaches are essentially the same. Augment the Z state with some other Markov state which determines the growth rate, and have it fluctuate within some bounds. Our Markov chain approach is much easier to solve. The backwards multiplicative jumps would work, but is very tricky since you would have agents jumping backwards over the endogenous threshold, leading to delay-differential equations.

It is important to note that we would prefer to only have a single Z state, and the L/H state is inconvenient rather than intrinsic to the economics. However, we have chosen this approach because it is the **simplest**.

Some Approaches that Do Not Work To summarize why some obvious candidates will fail the criteria established above in continuous time:

- *Poisson Arrival of Jumps Forward?* Consider an arrival of multiplicative jumps $\delta > 1$ so that after an arrival, Z becomes $\delta Z > Z$. The problem is the nature of Poisson arrivals. For any strictly positive interval of time, $\tau > 0$, the number of arrivals of jumps at intensity λ, $N \sim \text{Poisson}(\lambda\tau)$ with support $N = 0, \dots \infty$. With a continuum of agents, for any $\tau > 0$ there is some agent who gains an arbitrarily large N of arrivals, and then $Z(t+\tau) = \delta^N Z(t)$. As this holds for any τ and N, the support goes to infinity immediately with a continuum of agents, or very rapidly for a discrete but large number of agents.

- *Poisson Arrival of Jumps Forward with Decreasing Jump Sizes?* A variation is to have the δ jump decreasing in the relative Z—in a way similar to Acemoglu, Akcigit, and Celik (2014) However, if the jump sizes converge to 0, but are strictly positive, then no matter how quickly you have the jump sizes drop the support goes to infinity. Alternatively, you could have them actually reach 0 at a certain size, but that means arbitrarily choosing the maximum of support that the $\delta(Z)$ converges to, and ultimately requiring that the innovation process is a direct function of the distribution $\Phi(t, Z)$, as it requires $\delta(Z)$ dropping in relative terms. Both violations of our requirements, and if a resolution is possible, it would likely be much more complicated than our current setup.

- *Poisson Arrival of Drifts Rather than Jumps?* Instead of having jumps of δ, why not have Poisson arrivals of a controlled drift g, so that $\boldsymbol{\partial}_t Z(t) = gZ(t)$ with an arrival, and $\boldsymbol{\partial}_t Z(t) = 0$ otherwise? The problem is that in any positive time period, there are an arbitrarily large (but countable!) number of arrivals, but the amount of time is continuous (and uncountable). Hence, the firm spends measure 0 time growing, and the process is effectively deterministic.

- *IID growth rates?* As discussed, there are no IID continuous time processes that have positive quadratic variation and yet are measurable over time. This is a property of stochastic processes, and not specific to our model.[2] For example, take a discrete time model with iid draws of L and H growth rates each period. In the heuristic continuous time limit, this process ends up as deterministic growth at the long-run mean growth rate of the L and H.[3]

- *Singular Perturbation of a Markov Chain (i.e., asymptotically IID)?* The inconvenient part of our process is the augmented L/H state, which doesn't serve any purpose unto itself. Instead of actually writing down an IID discrete state, take the singular perturbation of a Markov chain for the discrete state (i.e., take the arrival rate of jumps between the states to infinity so that the current state of the Markov chain is asymptotically irrelevant for forecasting the future, but the process is never actually IID). This general approach is used for multi-scale stochastic volatility models, but using this for the drift (rather than rapidly switching volatility of Brownian motion, as those models use) doesn't help us as the Z ends up being asymptotically deterministic as it approaches the singularity for the same reason that the heuristic limit of the IID H/L discrete time process becomes deterministic. Scaling up the rate of growth in line with the speed of the switching would keep the process from being deterministic, but it would also require taking the support to infinity (i.e., becomes Brownian Motion in the limit).

- *Reflect Geometric Brownian Motion?* With an exogenously growing maximum of support of the distribution, we could simply have the stochastic process as reflected Brownian motion to keep it contained. But this approach breaks down when we want to endogenize the process. The first difficulty is in keeping the innovation independent of the $\Phi(t, Z)$ distribution, which is impossible since the reflection needs to happen at the current maximum of support. The bigger issue is endogenizing the growth rate of the frontier. The productivity of the highest Z is necessarily deterministic (or we would have to add an additional stochastic process for them) in this setup, but if everyone is reflected then the leader can never change. As the frontier agent never changes, this cuts off the interaction between innovation and technology diffusion in determining the frontier growth rate.[4]

[2] The intuition is that independence over time means that for any $\tau > 0$, the distribution of possible states for $Z(t+\tau)$ is independent of the current state $Z(t)$. Hence, the inf and sup of $Z(t+\tau)$ are the maximum and minimum of the stochastic shocks applied to $Z(t)$. But as this holds for any strictly positive $\tau > 0$, there is no way to integrate up time intervals with the inf and sup converging to each other with Riemann Sums (or the equivalent Lebesgue approach). The only way that this is measurable is if inf = sup for all τ, which means that the process is deterministic.

[3] Remember that the heuristic derivation of Brownian Motion from this sort of L/H tree also changes the time scale relative to the step size, which leads to infinite support.

[4] For the failure of this approach, it is easiest to think through the 3 firm scenario, where the 2 followers are arbitrarily bouncing off of the frontier agent.

- *Deterministic, Heterogeneous Growth Rates?* Assume there is heterogeneity in a growth state with doesn't change while the firm is operating (i.e., no switching). Clearly, the growth rate of the economy must be the largest of these heterogeneous growth rates. The issue is that those with the highest type will never fall back in relative terms, and hence never adopt any new technologies. If the types are permanent, then those firms will operate completely independent of the rest of the economy and essentially exist in autarky forever (with no interactions with the rest of the economy or each other). The alternative is that firms change types only when they adopt a new technology. The issue there is that the highest growth rate is an absorbing state, so eventually every firm will end up growing at that rate with no equilibrium technology diffusion. To get around this, you could add an arrival rate to fall back to a lower state and stay there until adopting again, but then you are using our Markov chain approach, just with $\lambda_\ell = 0$.

Of course, many of these issues are peculiar properties of continuous-time stochastic processes, rather than something intrinsic to our model. If written in discrete time (which is much more difficult to solve otherwise), the obvious solution is to have an iid L/H growth rates each period with control over growth while H. Due to the iid shocks, Z would be the only state of firms in the economy.

Appendix B Numerical Methods

The numerical methods are described in detail, as they are generally applicable to heterogeneous agent models. The structure is as follows: (1) Appendix B.1 collects all of the model equations; (2) Appendix B.2 summarizes the general approach; (3) Appendix B.3 explicitly maps all of the collected equations to be used by the algorithm; and (4) Appendix B.4 is just a variation of the mapping of equations using PDFs rather than CDFs for the KFEs.

B.1 Nested Summary of Equations for Endogenous Innovation

Summarizing the full set of equations to solve for $F_i(z)$ and $w_i(z)$ from Main Paper (23) to (26), (89), (97), (B.2), (B.7), (B.10) to (B.12), (B.14), (B.15), (B.22) to (B.26), (B.29), (B.30) and (B.34) to (B.36).

The Hamilton-Jacobi-Bellman Equation (HJBE) for $\kappa = 1$, or for $\psi = 1$ and $\kappa \neq 1$ is

$$0 = w_\ell(0) = w_h(0) \tag{B.1}$$

$$0 = 1 - (r + \lambda_\ell + \eta - (1-\psi)gF'(0))w_\ell(z) - gw_\ell'(z) + \lambda_\ell w_h(z) \tag{B.2}$$

$$0 = 1 - (r + \lambda_h + \eta)w_h(z) - \left(g - \tfrac{\chi}{2}w_h(z)\right)w_h'(z) + \left(\lambda_h + (1-\psi)gF'(0)\right)w_\ell(z) + \tfrac{\chi}{4}w_h(z)^2 \tag{B.3}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$, use Main Paper (97) to find the integro-differential equations for the HJBE,

$$\begin{aligned} 0 = {} & 1 - (r + \lambda_\ell + \eta - (1-\psi)\kappa g F'(0)F(z)^{\kappa-1})w_\ell(z) - gw_\ell'(z) + \lambda_\ell w_h(z) \\ & + (1-\psi)\kappa(\kappa-1)gF'(0)F(z)^{\kappa-2}F'(z)e^{-z}\int_0^z w_\ell(\hat{z})\mathrm{d}\hat{z} \end{aligned} \tag{B.4}$$

$$\begin{aligned} 0 = {} & 1 - (r + \lambda_h + \eta)w_h(z) - \left(g - \tfrac{\chi}{2}w_h(z)\right)w_h'(z) + \left(\lambda_h + (1-\psi)\kappa g F'(0)F(z)^{\kappa-1}\right)w_\ell(z) + \tfrac{\chi}{4}w_h(z)^2 \\ & + (1-\psi)\kappa(\kappa-1)gF'(0)F(z)^{\kappa-2}F'(z)e^{-z}\int_0^z w_\ell(\hat{z})\mathrm{d}\hat{z} \end{aligned} \tag{B.5}$$

The innovation rate is,

$$\gamma(z) = \tfrac{\chi}{2} w_h(z) \tag{B.6}$$

In either case, $\bar{z}$ and g are related through,

$$g \equiv \gamma(\bar{z}) \tag{B.7}$$

The KFE can be solved as an initial value problem (or boundary value problem depending on the algorithm) subject to value matching,

$$0 = F_\ell(0) = F_h(0) \tag{B.8}$$
$$0 = gF_\ell'(z) + \lambda_h F_h(z) - (\lambda_\ell + \eta)F_\ell(z) + (1-\theta)gF'(0)F(z)^\kappa - gF_\ell'(0) \tag{B.9}$$
$$0 = (g - \gamma(z))F_h'(z) + \lambda_\ell F_\ell(z) - (\lambda_h + \eta)F_h(z) - gF_h'(0) \tag{B.10}$$
$$0 = -\frac{1}{\psi}\left(\zeta + \tfrac{1}{\vartheta}\theta^2 + \tfrac{1}{\varsigma}\kappa^2\right) + \int_0^{\bar{z}} e^z w_\ell(z)\mathrm{d}z - (1-\theta)\int_0^{\bar{z}} e^z w_\ell(z)F(z)^\kappa \mathrm{d}z \tag{B.11}$$

Note that if $\kappa = 1$, the KFE is a linear system of ODEs. More generally, the endogenous κ must solve the implicit equation,

$$\kappa = \frac{-\varsigma\psi(1-\theta)}{2}\int_0^{\bar{z}} e^z w_\ell(z)\log(F(z))F(z)^\kappa \mathrm{d}z \tag{B.12}$$

Similarly, the following determines the endogenous θ equation

$$\theta = \frac{\psi\vartheta}{2}\int_0^{\bar{z}} e^z w_\ell(z)F(z)^\kappa \mathrm{d}z \tag{B.13}$$

KFE in PDFs with $\kappa = 1$ Take (B.9) and (B.10) and differentiate to find the KFE in the PDFs when $\kappa = 1$ and $f_i(z) \equiv \boldsymbol{\partial}_z F_i(z)$,

$$0 = gf_\ell'(z) + \lambda_h f_h(z) - (\lambda_\ell + \eta)f_\ell(z) + (1-\theta)gf(0)f(z) \tag{B.14}$$
$$0 = (g - \gamma(z))f_h'(z) + \lambda_\ell f_\ell(z) - (\lambda_h + \eta + \gamma'(z))f_h(z) \tag{B.15}$$

Verification and Guesses In the general case, given a $F'(0)$, one could solve the quadratic in (B.16) and (B.17) for c_ℓ and c_h and then use (B.18) to find the maximum g as $\bar{z} \to \infty$,[5]

$$0 = 1 - (r + \lambda_\ell + \eta - (1-\psi)\tfrac{\chi}{2}c_h F'(0))c_\ell + \lambda_\ell c_h \tag{B.16}$$
$$0 = 1 - (r + \lambda_h + \eta)c_h + \left(\lambda_h + (1-\psi)\tfrac{\chi}{2}c_h F'(0)\right)c_\ell + \tfrac{\chi}{4}c_h^2 \tag{B.17}$$
$$g_{\max} = \tfrac{\chi}{2}c_h \tag{B.18}$$

In the case of $\psi = 1$, then the maximum g is independent of $F'(0)$ and comes from

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell} \tag{B.19}$$
$$g < \bar{\lambda}(r+\eta)\left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r+\eta)^2}}\right] \tag{B.20}$$

[5]Since we have put these equations into the general collocation system, it is not necessary to solve this for a particular $\bar{z}$. To find $g_{\max}$ numerically, just choose a large $\bar{z}$.

While not necessary to solve for the equilibrium, the non-stationary value functions can be calculated with

$$v(0) \equiv \frac{1+\eta\int_0^{\bar{z}} e^z w_\ell(z)\mathrm{d}z}{r-g} \tag{B.21}$$

$$v_i(z) = v(0) + \int_0^z e^{\hat{z}} w_i(\hat{z})\mathrm{d}\hat{z} \tag{B.22}$$

The mass of high and low firms fulfills,

$$1 = F_\ell(\bar{z}) + F_h(\bar{z}) \tag{B.23}$$

$$F_\ell(\bar{z}) = \frac{1}{\lambda_\ell+\lambda_h+\eta}\left(gF_h'(0)+\eta+\lambda_h\right) \tag{B.24}$$

$$F_h(\bar{z}) = \frac{1}{\lambda_\ell+\lambda_h+\eta}\left(-gF_h'(0)+\lambda_\ell\right) \tag{B.25}$$

Note, from (B.23) and (B.25) that the maximum guess on $F_h'(0)$ is,

$$F_h'(0) < \frac{\lambda_\ell}{g} \tag{B.26}$$

An alternative approach for verifying guesses is to start with $F_h(\bar{z}) \in (0,1)$ and use

$$F_h'(0) = \frac{\lambda_\ell - (\eta+\lambda_h+\lambda_\ell)F_h(\bar{z})}{g} \tag{B.27}$$

From the properties of the solution, we also know the following constraints,

$$w_i(z) \geq 0 \tag{B.28}$$

$$w_i'(z) \geq 0 \tag{B.29}$$

And to be a valid CDF,

$$F_i(z) \geq 0 \tag{B.30}$$

$$F_i'(z) \geq 0 \tag{B.31}$$

B.2 Summary of Spectral Solution Method

This section summaries the main algorithm, the key trick, and why other methods were less successful.

Why not use Finite-Differences? One option is to solve the ODEs using methods such as those in Achdou, Lasry, Lions, and Moll (2014), with an outside iteration to find the equilibrium conditions? While finite difference solutions to a single ODE are almost always faster than spectral methods for non-stiff systems without singularities, they don't seem to work well here for several reasons:

1. the ODEs for the HJBE and KFE have a stronger degree of coupling than many models, especially when $\psi < 1$. This would lead to an iterative algorithm nesting the choice a fairly large number of 'parameters' to solve the ODEs with, then checking the residual of the ODEs in the equilibrium conditions and iterating. The spectral methods here jointly solve all the equations at the same time;

2. the ODEs here appear to be stiff, and—more importantly—have a singular point at $\bar{z}$ where $\lim_{z\to\bar{z}}(g - \gamma(z)) = 0$. This means that the ODEs for $w_h(z)$ and $F_h(z)$ have a singular point at $z = \bar{z}$ where the coefficient in front of the $w_h'(z)$ and $F_h'(z)$ terms are 0;

3. because the HJBE here appears to be stiff, it requires different finite-difference schemes, and where application of a downwind method such as in Achdou, Lasry, Lions, and Moll (2014) without an adjustment seems to fail;

4. the stiffness (and, perhaps, singularity) leads to problems of stability, not just accuracy For example, the only finite difference schemes guaranteed to maintain the monotonicity of the function (preserving shape here) are the first-order backwards Euler, which is equivalent to downwind in this setup. However, this is not an appropriate for stiff systems.

Some of these problems, such as the singularity, may be particular to endogenous growth models, but the coupling and stiffness may apply to a large number of models. In general, the method proposed here is easy to implement, especially when a large number of equilibrium conditions must be kept, and can be used as a first attempt prior to looking for a specialized finite-difference approach.

Differential-Algebraic Equation? Note that solving the HJBE for the γ choice leads to a differential-algebraic equation (DAE) with the first-order condition and the corresponding bellman equation. We substitute in the FOC to the Bellman equation to form a nonlinear Bellman equation, whereas it may be preferable to solve the model as a DAE without the substitution. While our approach worked sufficiently well in our case, it may be preferable to split (B.3) into the original bellman, conditional on $\gamma(z)$, and the $\gamma(z)$ equation from the first-order condition (which, in turn, depends on the $w_i(z)$). This would eliminate the nonlinearity in the $w_h(z)$ ODE. The collocation method to solve it would be otherwise identical in principle after adding a function approximation for the $\gamma(z)$ and the additional equations.

Spectral Method The general structure of the problem is a set of ODEs with parameters constrained by equilibrium conditions (themselves functions of the solutions to the ODEs). A natural approach is to approximation the $w_i(z)$ and $F_i(z)$ functions some finite dimensional polynomial, and find the coefficients that fulfill all of the ODEs and equilibrium conditions—i.e., every equation in Appendix B.1. This general approach is called a spectral method, and we will concentrate on the spectral collocation method (i.e., solve a nonlinear system in the coefficients on the function basis with a matching number of equations). Alternatively, this could be done with the related Weighted Residual methods, over-constraining the system of equations and finding the minimum residual using nonlinear least squares, etc.[6] To summarize the approach used:

1. Pick a $\bar{z}$. In the case of unbounded support, this should be a large number to ensure convergence. In the case of bounded support, these will parameterize the set of stationary equilibrium (though there would not be a feasible for every $\bar{z}$).

[6]In the special case of $\psi = 1$ with g known, the problems are decoupled. In that case, the HJBE can be solved with a stiff ODE solver such as Matlab's `ode15s` as long as the solution terminates as it approaches the singularity. To accomplish this, stop at the $\bar{z}$ such that $g - \frac{\chi}{2}w_h(\bar{z}) \approx 0$. For stability choose a small threshold value, but ensure that at the chosen $\bar{z}$, $\frac{\chi}{2}w_h(\bar{z}) < g$, which will ensure that singularities don't occur in subsequent calculations and cause havoc or divergence. In matlab, see ODE settings of 'Events'. See `http://www.mathworks.com/help/matlab/ref/odeset.html#f92-1017470`.

2. Choose a Chebyshev basis for the $w_i(z)$ and $F_i(z)$ adapted to $z \in [0, \bar{z}]$.[7]

3. Calculate the collocation nodes (i.e., the roots of the basis polynomials) and the quadrature weights for the particular basis. For example, if using Chebyshev basis, then the nodes would be the Gauss-Chebyshev quadrature weights.

   Aligning the quadrature and collocation nodes is the key step.

4. Setup a system of all the equations, ODEs, and boundary conditions

   - For the fixed $\bar{z}$, the set of equations are for $\{w_\ell(z), w_h(z), F_\ell(z), F_h(z), \kappa, \theta\}$ (where the functions are parameterized by a finite dimensional set of coefficients)
   - The complete set of equations in Appendix B.1 must hold, where the functional equations must hold at all of the collocation node points.
   - A key trick here is that we are evaluating the $w_i(z)$ and $F_i(z)$ at the same nodes as we would use for the associated quadrature. This means that with the appropriate Gaussian quadrature weights, we can calculate integrals such as $\int_0^{\bar{z}} w_\ell(z)(1 - F(z))\mathrm{d}z$ as a simple linear function. While this general approach significantly simplifies the calculations, its main purpose is to enable auto-differentiation for solvers to use the Jacobian of the residual.

5. Use a nonlinear solver to find the coefficients of the $w_i(z)$ and $F_i(z)$, along with any other variables such as g, κ, or θ.

   - To solve systems of this size, it is necessary to find the Jacobian of the system of equations. This can be accomplished easily using auto-differentiation. Furthermore, the Jacobian is also somewhat sparse, so specialized algorithms that exploit sparsity are helpful.
   - Since the number of variables is large (e.g., if 100 nodes are used per function basis, this is 402 variables), a good solver is necessary. We are using the NLSSOL constrained nonlinear least squares solver from `http://tomopt.com/tomlab/products/npsol/solvers/NLSSOL.php`, which has built in auto-differentiation. With this, we can find the complete solution in $\approx$ 2-20 seconds with 100 nodes per function.

B.3 Joint Spectral Collocation Algorithm

Notation For a vectors x, y and scalar a: Denote the scalar-vector products as ax, the dot-product (or matrix-vector product) as $x \cdot y$, the point-wise product of vectors as $x \odot y$. For the j'th element denote it $x(j)$ and a slice between j and k as $x(j:k)$ (where the first index is 0).

For vectors that are only on the interior nodes (i.e., for quadrature), these are denoted such as $e^{\vec{z}}_{\text{int}} \equiv e^{\vec{z}}(1:N)$ or $\vec{z}_{\text{int}} \equiv \vec{z}(1:N)$

[7] An alternative consideration for the unbounded case is a Gauss-Laguerre basis. For relatively large $\bar{z}$, the benefit of Gauss-Laguerre is that the quadrature nodes will calculate $\int_0^\infty h(z)\mathrm{d}z$ rather than using $\int_0^{\bar{z}} h(z)\mathrm{d}z$ to approximate the integral. Furthermore, the nodes of the polynomial are spaced closer to the minimum of support. The worry is that since $\bar{z}$ is finite, for the ODE calculations issues such as Runge's Phenomenon in the right tail are not minimized—while Chebyshev polynomials provide the best convergence for these artifacts at corners. Hence, to minimize errors and nest the unbounded and bounded cases, we chose Chebyshev polynomials. While we see some oscillations at the corners, with Chebyshev basis the errors seem to be within tolerance bounds.

Parameters The complete set of model parameters for the nested model is $\{r, \lambda_\ell, \lambda_h, \eta, \psi, \chi, \zeta, \vartheta, \varsigma\}$, and if they are not endogenous, then an additional $\left\{\bar{\theta}, \bar{\kappa}\right\}$ is used (and the corresponding $\{\vartheta, \varsigma\}$ are not necessary). If the consumer's IES is used for discounting, then r is determined in equilibrium from ρ and Λ.

In addition, $\bar{z}$ is effectively a fixed parameter for the calculations. It is set large (e.g., $\bar{z} = 9.3$, which corresponds to $\bar{Z}(t)/M(t) \approx 10000$) in the case of unbounded support, and is set to a fixed number when looking for the set of equilibria for bounded support.

Setup and Function Approximation Given a $\bar{z}$, define the approximations with $N-1$ order Chebyshev polynomials, $T_k(z)$ adapted to the support $[0, \bar{z}]$. Then, approximation $w_i(z)$ and $F_i(z)$ through,

$$w_i(z) \approx \sum_{n=0}^{N-1} c_{\text{in}} T_n(z) \tag{B.32}$$

$$F_i(z) \approx \sum_{n=0}^{N-1} d_{\text{in}} T_n(z) \tag{B.33}$$

Denote the vectors of coefficients as $d_i \in \mathbb{R}^N$ and $c_i \in \mathbb{R}^N$. Define the Chebyshev polynomial roots,

$$\vec{z}_{\text{int}} \equiv \{z_1, \dots z_N\} \in \mathbb{R}^N \tag{B.34}$$

And the complete set of nodes including boundary values as (with $z_0 \equiv 0$ and $z_{N+1} \equiv \bar{z}$)

$$\vec{z} \equiv \{0, z_1, \dots z_N, z_N, \bar{z}\} \in \mathbb{R}^{N+2} \tag{B.35}$$

Define the basis matrices for these nodes, B, by stacking stacking the polynomials evaluated at $\vec{z}$, Similarly, define B' as the basis of the first derivative, and $\breve{B}$ as the basis of the partial integrals.[8]

$$B \equiv \begin{bmatrix} T_0(z_0) & \dots & T_{N-1}(z_0) \\ \dots & & \dots \\ T_0(z_{N+1}) & \dots & T_{N-1}(z_{N+1}) \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{B.37}$$

$$B' \equiv \begin{bmatrix} T_0'(z_0) & \dots & T_{N-1}'(z_0) \\ \dots & & \dots \\ T_0'(z_{N+1}) & \dots & T_{N-1}'(z_{N+1}) \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{B.38}$$

$$\breve{B} \equiv \begin{bmatrix} \int_0^{z_0} T_0(z)\mathrm{d}z & \dots & \int_0^{z_0} T_{N-1}(z)\mathrm{d}z \\ \dots & & \dots \\ \int_0^{z_{N+1}} T_0(z)\mathrm{d}z & \dots & \int_0^{z_{N+1}} T_{N-1}(z)\mathrm{d}z \end{bmatrix} \in \mathbb{R}^{(N+2)\times N} \tag{B.39}$$

[8]With `compecon`, the basis can be calculated for nodes `all_nodes` and function space `fspace` as `B = funbas(fspace, vec_z_all, 0)`. The basis of the derivative is simply `B_p = funbas(fspace, vec_z_all, 1)`. With compecon, to calculate the quadrature weights for Chebyshev quadrature use `qnwcheb(N,0,z_bar)`. The basis for the integrals doesn't appear to be in compecon, but can be calculated using the recurrence formulas for Chebyshev polynomials (`https://en.wikipedia.org/wiki/Chebyshev_polynomials#Differentiation_and_integration`). For the $\tilde{T}_n(x)$ polynomials on $[-1, 1]$, the formula is $\int \tilde{T}_n(x)\mathrm{d}x = \frac{1}{2}\left(\frac{\tilde{T}_{n+1}(x)}{n+1} - \frac{\tilde{T}_{n-1}(x)}{n-1}\right)$. When adapted to $[0, \bar{z}]$, this becomes

$$\int_0^z T_n(\tilde{z})\mathrm{d}\tilde{z} = \frac{\bar{z}}{2}\left(\frac{T_{n+1}(z)}{n+1} - \frac{T_{n-1}(z)}{n-1}\right) - \frac{\bar{z}}{2}\left(\frac{T_{n+1}(0)}{n+1} - \frac{T_{n-1}(0)}{n-1}\right) \tag{B.36}$$

where for $z_j \in \vec{z}$ the $T_n(z_j)$ can be calculated from B.

With these, the functions and derivatives evaluated at all $\vec{z}$ are a simple matrix-vector product,

$$\vec{w}_i \equiv \{w_i(z_n)\}_{n=0}^{N+1} = B \cdot c_i \in \mathbb{R}^{N+2} \tag{B.40}$$
$$\vec{w}'_i \equiv \{w'_i(z_n)\}_{n=0}^{N+1} = B' \cdot c_i \in \mathbb{R}^{N+2} \tag{B.41}$$
$$\breve{w}_i \equiv \left\{\int_0^{z_n} w_i(z)\mathrm{d}z\right\}_{n=0}^{N+1} = \breve{B} \cdot c_i \in \mathbb{R}^{N+2} \tag{B.42}$$
$$\vec{F}_i \equiv \{F_i(z_n)\}_{n=0}^{N+1} = B \cdot d_i \in \mathbb{R}^{N+2} \tag{B.43}$$
$$\vec{F}'_i \equiv \{F'_i(z_n)\}_{n=0}^{N+1} = B' \cdot d_i \in \mathbb{R}^{N+2} \tag{B.44}$$

Given Chebyshev quadrature weights $\omega \in \mathbb{R}^N$ with $\vec{h}_{\text{int}} \equiv \{h(z)|z \in \vec{z}_{\text{int}}\} \in \mathbb{R}^N$, integrals over the entire domain are a simple vector product for any $h(z)$,[9]

$$\int_0^{\bar{z}} h(z)\mathrm{d}z \approx \omega \cdot \vec{h}_{\text{int}} \tag{B.45}$$

Approximate Problem With the finite dimensional approximation, the complete set of guesses to calculate a root to the system of equations is,

$$x \equiv \{c_0, c_1, \ldots c_{N-1}, d_0, d_1, \ldots d_{N-1}, \kappa, \theta\} \in \mathbb{R}^{4N+2} \tag{B.46}$$

Define a residual operator $\mathcal{L} : \mathbb{R}^{4N+2} \to \mathbb{R}^{\hat{N}}$. The solution is an x^* such that $\mathcal{L}(x^*) \approx 0$. If this was square, then $\hat{N} = 4N + 2$. Add linear inequality and equality constraints with matrix $\Psi \in \mathbb{R}^{\hat{N}_{\text{con}} \times (4N+2)}$ and vectors $\bar{b}, \underline{b}$ in $\mathbb{R}^{\hat{N}_{\text{con}}}$ and box-bounds $\bar{x}, \underline{x} \in \mathbb{R}^{4N+2}$,

$$\begin{aligned} x^* = \arg\min_x &\left\{\frac{1}{2}\mathcal{L}(x) \cdot \mathcal{L}(x)\right\} \\ \text{s.t.}\ &\underline{b} \leq \Psi \cdot x \leq \bar{b} \\ &\underline{x} \leq x \leq \bar{x} \end{aligned} \tag{B.47}$$

Summary of Pre-calculations At the end of the pre-calculations, we have a $\{\vec{z}, B, B', \omega, \Omega\}$ to be used in further calculations. Insofar as there is a "trick" to the algorithm, it is that the quadrature and collocation schemes are chosen so the nodes line up, which means that the functions only need to be calculated at a single set of nodes for both the ODE and integral equilibrium conditions.

Calculating the Residual Given the x and using the fixed B, B', and $\vec{z}$, calculate the following:

Setup in $\mathcal{L}$ Calculate $\vec{F}_\ell, \vec{F}_h, \vec{w}_\ell, \vec{w}_h, \breve{w}_\ell, \vec{F}'_\ell, \vec{F}'_h, \vec{w}'_\ell, \vec{w}'_h$, and (B.6) and (B.7),

$$F'(0) = \vec{F}'_\ell(0) + \vec{F}'_h(0) \in \mathbb{R} \tag{B.48}$$
$$\vec{\gamma} = \tfrac{\chi}{2}\vec{w}_h \in \mathbb{R}^{N+2} \tag{B.49}$$
$$g = \vec{\gamma}(N+1) \in \mathbb{R} \tag{B.50}$$
$$r = \rho + \Lambda g \in \mathbb{R}, \quad \text{if using consumer with CRRA preferences} \tag{B.51}$$
$$\vec{F}^\kappa = (\vec{F}_\ell + \vec{F}_h)^\kappa \in \mathbb{R}^{N+2}, \quad \text{i.e., pointwise power.} \tag{B.52}$$

The following are only defined for interior nodes,

$$\tilde{w}_{\ell,\text{int}} = e^{\vec{z}}_{\text{int}} \odot \vec{w}_{\ell,\text{int}} \in \mathbb{R}^N \tag{B.53}$$

[9] Note that the end-points are not used in the Gaussian quadrature formula. To calculate the quadrature weights for Chebyshev quadrature in compecon use `qnwcheb(N - 1,0,z_bar)`

Equation for ℓ HJBE From (B.2). This provides a vectorized calculation of all residuals.

$$0 \approx 1 - (r + \lambda_\ell + \eta - (1-\psi) g F'(0))\vec{w}_\ell - g\vec{w}'_\ell + \lambda_\ell \vec{w}_h \tag{B.54}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$, from (B.4),

$$\begin{aligned} 0 &\approx 1 - (r + \lambda_\ell + \eta - (1-\psi)\kappa g F'(0)\, \vec{F}^{\kappa-1}) \odot \vec{w}_\ell - g\vec{w}'_\ell + \lambda_\ell \vec{w}_h \\ &\quad + (1-\psi)\kappa(\kappa-1) g F'(0)\, \vec{F}^{\kappa-2} \odot (\vec{F}'_\ell + \vec{F}'_h) \odot e^{-\vec{z}} \odot \breve{w}_\ell \end{aligned} \tag{B.55}$$

Equation for h HJBE From (B.3). As discussed in Appendix B.2, it may be preferable to split the $\vec{\gamma}$ and $\vec{w}_h$ to create a DAE, and add in the equations separately.

$$0 \approx 1 - (r + \lambda_h + \eta)\vec{w}_h - (g - \vec{\gamma}) \odot \vec{w}'_h + \left(\lambda_h + (1-\psi) g F'(0)\right) \vec{w}_\ell + \tfrac{\chi}{4} \vec{w}_h \odot \vec{w}_h \tag{B.56}$$

In the general case of $\kappa \neq 1$ and $\psi < 1$, from (B.5),

$$\begin{aligned} 0 &= 1 - (r + \lambda_h + \eta)\vec{w}_h - (g - \vec{\gamma}) \odot \vec{w}'_h + \left(\lambda_h + (1-\psi)\kappa g F'(0) \vec{F}^{\kappa-1}\right) \odot \vec{w}_\ell + \tfrac{\chi}{4} \vec{w}_h \odot \vec{w}_h \\ &\quad + (1-\psi)\kappa(\kappa-1) g F'(0)\, \vec{F}^{\kappa-2} \odot (\vec{F}'_\ell + \vec{F}'_h) \odot e^{-\vec{z}} \odot \breve{w}_\ell \end{aligned} \tag{B.57}$$

Equation for ℓ KFE From (B.9)

$$0 \approx g\vec{F}'_\ell + \lambda_h \vec{F}_h - (\lambda_\ell + \eta)\vec{F}_\ell + (1-\theta) g F'(0) \vec{F}^{\kappa} - g F'_\ell(0) \tag{B.58}$$

Equation for h KFE From (B.10)

$$0 \approx (g - \vec{\gamma}) \odot \vec{F}'_h + \lambda_\ell \vec{F}_\ell - (\lambda_h + \eta)\vec{F}_h - g F'_h(0) \tag{B.59}$$

Equations for initial conditions From (B.1) and (B.8),

$$0 \approx F_\ell(0) \tag{B.60}$$
$$0 \approx F_h(0) \tag{B.61}$$
$$0 \approx w_\ell(0) \tag{B.62}$$
$$0 \approx w_h(0) \tag{B.63}$$

Equation for Value Matching Since we are calculating for a finite $\bar{z}$, the general case of (B.11) nests the unbounded support for a 'large' $\bar{z}$. In the case of exogenously given θ or κ, then let ς and/or $\vartheta = \infty$ to ensure they don't add to the cost.

$$0 \approx -\frac{1}{\psi}\left(\zeta + \tfrac{1}{\vartheta}\theta^2 + \tfrac{1}{\varsigma}\kappa^2\right) + \omega \cdot \tilde{w}_{\ell,\text{int}} - (1-\theta)\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] \tag{B.64}$$

Equation for Endogenous κ From (B.12),

$$0 = -\kappa + \begin{cases} \frac{-\varsigma\psi(1-\theta)}{2}\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \log \vec{F}_{\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] & \text{if endogenous} \\ \bar{\kappa} & \text{if fixed at } \bar{\kappa} \end{cases} \tag{B.65}$$

Equation for Endogenous θ From (B.13),

$$0 \approx -\theta + \begin{cases} \frac{\psi\vartheta}{2}\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \vec{F}^{\kappa}_{\text{int}}\right] & \text{if endogenous} \\ \bar{\theta} & \text{if fixed at } \bar{\theta} \end{cases} \tag{B.66}$$

Equation for $F(\bar{z}) = 1$ From (B.23)

$$0 \approx F_l(\bar{z}) + F_h(\bar{z}) - 1 \tag{B.67}$$

Linear Constraints The linear equality constraints, such as (B.23) have all been added directly to the $\mathcal{L}$ residual. Linear inequality constraints come from (B.27), (B.28), (B.30) and (B.31).[10] As this polynomial basis is not shape preserving, the constraints on $F_i(z)$ and $F_i'(z) > 0$ are unlikely to hold everywhere. Instead, choose a small threshold $\epsilon > 0$ such that $F_i'(z) > -\epsilon$ for all z and $F_i'(z) > -\epsilon$ for all z. The oscillations around endpoints, while minimized with a Chebyshev basis, could cause problems in our algorithm since g is so dependent on $F'(0)$, so $F_i'(0) \geq 0$ are added directly as a special case. While the $w_i(z) > 0$ and $F_i(z) \geq 0$ constraints could also be added with appropriate ϵ, they are left out for simplicity.

Recall that $w_i(0) \approx \bar{Z}(0,:) \cdot c_i$, $F_i(0) \approx \bar{Z}(0,:) \cdot d_i$, etc. Let $\vec{0} \equiv 0_N, \vec{\epsilon} \equiv \epsilon_{N+2}, \vec{\infty} \equiv \infty_N$ and $\vec{\mathbf{0}} \equiv 0_{(N+2)\times N}$

$$\overbrace{\begin{bmatrix} -\vec{\epsilon} \\ -\vec{\epsilon} \\ 0 \\ 0 \end{bmatrix}}^{\underline{b}} \leq \overbrace{\begin{bmatrix} \vec{\mathbf{0}} & \vec{\mathbf{0}} & B' & \vec{\mathbf{0}} & \vec{0} & \vec{0} \\ \vec{\mathbf{0}} & \vec{\mathbf{0}} & \vec{\mathbf{0}} & B' & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & B'(0,:) & \vec{0} & 0 & 0 \\ \vec{0} & \vec{0} & \vec{0} & B'(0,:) & 0 & 0 \end{bmatrix}}^{\Psi} x \leq \overbrace{\begin{bmatrix} \vec{\infty} \\ \vec{\infty} \\ \infty \\ \infty \end{bmatrix}}^{\bar{b}}, \text{ from } \begin{bmatrix} (\text{B.31}) \\ (\text{B.31}) \\ (\text{B.31}) \\ (\text{B.31}) \end{bmatrix} \tag{B.68}$$

With bounds on the x as,

$$\overbrace{\begin{bmatrix} -\vec{\infty} \\ -\vec{\infty} \\ -\vec{\infty} \\ -\vec{\infty} \\ 0 \\ 0 \end{bmatrix}}^{\underline{x}} \leq x \leq \overbrace{\begin{bmatrix} \vec{\infty} \\ \vec{\infty} \\ \vec{\infty} \\ \vec{\infty} \\ \infty \\ 1 \end{bmatrix}}^{\bar{x}} \tag{B.69}$$

Mean and Gini Given the Chebyshev collocation nodes and the solution for d_i, the mean and Gini coefficient can be calculated with,

$$\text{Mean} = \int_0^{\bar{z}} (1 - F(z))\mathrm{d}z = \omega \cdot (1 - \vec{F}_{\text{int}}) \tag{B.70}$$

$$\text{Gini} = 1 - \frac{\int_0^{\bar{z}} ((1 - F(z))^2 \mathrm{d}z}{\int_0^{\bar{z}} (1 - F(z))\mathrm{d}z} = 1 - \frac{\omega \cdot (1 - \vec{F}_{\text{int}})^2}{\omega \cdot (1 - \vec{F}_{\text{int}})} \tag{B.71}$$

B.4 Joint Spectral Collocation Algorithm for $\kappa = 1$ in PDFs

In the case of $\kappa = 1$ with no endogeneity in θ or κ, the solution may be more stable if solved in PDFs, $f_i(z) \equiv \boldsymbol{\partial}_z F_i(z)$. The algorithm is essentially the same as in Appendix B.3, but the KFE residuals and equilibrium conditions are changed to use the basis in $f_i(z)$,

Differences in Setup The d_n vector of coefficients now represents $f_i(z)$,

$$f_i(z) \approx \sum_{n=0}^{N-1} d_{\text{in}} T_n(z) \tag{B.72}$$

[10] In an interior solution, few of the inequality constraints would be binding and $\epsilon \approx 0$.

Where we can use the existing basis B and B' to find

$$\vec{f_i} \equiv \{f_i(z_n)\}_{n=0}^{N+1} = B \cdot d_i \in \mathbb{R}^{N+2} \tag{B.73}$$

$$\vec{f} = \vec{f_\ell} + \vec{f_h} \tag{B.74}$$

$$\vec{f_i'} \equiv \{f_i'(z_n)\}_{n=0}^{N+1} = B' \cdot d_i \in \mathbb{R}^{N+2} \tag{B.75}$$

To find $F_i(z)$, we need to calculate the integral of $f_i(z)$ at each of the nodes,

$$\vec{F} \equiv \left\{\int_0^{z_n} (f_\ell(z) + f_h(z))\mathrm{d}z\right\}_{n=0}^{N+1} \tag{B.76}$$

$$= \breve{B} \cdot (d_\ell + d_h) \in \mathbb{R}^{N+2} \tag{B.77}$$

Calculating the Residual Given the x and using the fixed B, B', and $\vec{z}$, calculate the following:
Setup in $\mathcal{L}$ Calculate $\vec{f_\ell}, \vec{f_h}, \vec{w}_\ell, \vec{w}_h, \vec{f_\ell'}, \vec{f_h'}, \vec{F}, \vec{w}_\ell', \vec{w}_h'$, and (B.6) and (B.7),

$$f(0) = \vec{f_\ell}(0) + \vec{f_h}(0) \in \mathbb{R} \tag{B.78}$$

$$\vec{\gamma} = \tfrac{\chi}{2}\vec{w}_h \in \mathbb{R}^{N+2} \tag{B.79}$$

$$g = \vec{\gamma}(N+1) \in \mathbb{R} \tag{B.80}$$

$$r = \rho + \Lambda g \in \mathbb{R}, \quad \text{if using consumer with CRRA preferences} \tag{B.81}$$

The following are only defined for interior nodes,

$$\tilde{w}_{\ell,\text{int}} = e^{\vec{z}}_{\text{int}} \odot \vec{w}_{\ell,\text{int}} \in \mathbb{R}^N \tag{B.82}$$

Equation for ℓ HJBE From (B.2). This provides a vectorized calculation of all residuals.

$$0 \approx 1 - (r + \lambda_\ell + \eta - (1-\psi)gf(0))\vec{w}_\ell - g\vec{w}_\ell' + \lambda_\ell \vec{w}_h \tag{B.83}$$

Equation for h HJBE From (B.3). Again, consider a split to a DAE as an alternative.

$$0 \approx 1 - (r + \lambda_h + \eta)\vec{w}_h - (g - \vec{\gamma}) \odot \vec{w}_h' + (\lambda_h + (1-\psi)gf(0))\, \vec{w}_\ell + \tfrac{\chi}{4}\vec{w}_h \odot \vec{w}_h \tag{B.84}$$

Equation for ℓ KFE From (B.14)

$$0 \approx g\vec{f_\ell'} + (\lambda_h + (1-\theta)gf(0))\vec{f_h} - (\lambda_\ell + \eta - (1-\theta)gf(0))\vec{f_\ell} \tag{B.85}$$

Equation for h KFE From (B.10)

$$0 \approx \left(g - \frac{\chi}{2}\vec{w}_h\right) \odot \vec{f_h'} + \lambda_\ell \vec{f_\ell} - \left(\lambda_h + \eta + \frac{\chi}{2}\vec{w}_h'\right) \odot \vec{f_h} \tag{B.86}$$

Equations for initial conditions From (B.1),

$$0 \approx w_\ell(0) \tag{B.87}$$

$$0 \approx w_h(0) \tag{B.88}$$

Equation for Value Matching Since we are calculating for a finite $\bar{z}$, the general case of (B.11) nests the unbounded support for a 'large' $\bar{z}$.

$$0 \approx -\frac{\zeta}{\psi} + \omega \cdot \tilde{w}_{\ell,\text{int}} - (1-\theta)\,\omega \cdot \left[\tilde{w}_{\ell,\text{int}} \odot \vec{F}_{\text{int}}\right] \tag{B.89}$$

Equation for $F(\bar{z}) = 1$ From (B.23)

$$0 \approx \vec{F}(N+1) - 1 \tag{B.90}$$

Appendix C More Results and Proofs for H/L Innovation Process

This section collects a additional proofs and derivations for our main model.

C.1 Limit of Draw Arrival Process

The following is a rough, heuristic derivation of the law of motion and cost function for searching which yields a conditional draw above the firm's search threshold.

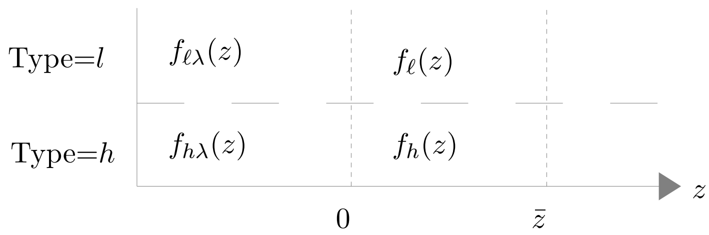

Figure 1: Search Regions PDF Before Taking Limit

Instead of instantaneous draws, assume a firm choosing to adopt has an arrival rate of $\bar{\lambda} > 0$ of opportunities. While attempting to adopt, they pay a normalized flow cost of $\zeta\bar{\lambda}$. Note that we are scaling the flow cost by the arrival rate in order to take the limit and have a finite expected value of search costs.

Furthermore, assume that the firm draws *unconditionally* from the z distribution in the economy (rather than simply those above their current threshold), and starts with a ℓ type. In the stationary equilibrium, as all agents start low, searching firms accept a draw if they get above the normalized cutoff of 0. The proof will construct a limit where agents get a successful draw above 0 in any infinitesimal time period, and hence draw from the conditional distribution of $z \geq 0$.

Define $F_{\ell\lambda}(z)$ and $F_{h\lambda}(z)$ to be the CDFs of agents in the $z < 0$ region, as in Figure 1, which will be shown to disappear in the limit. See Main Paper Appendix A.2 for a proof that the thresholds are identical for both types.

As firms in the region $F_{\ell\lambda}(z)$ are otherwise identical, set the mass of searching agents as $F_{\ell\lambda}(0)$. Assume that agents have an unconditional draw of all z within the economy, then conditional on a draw, the probability of escaping the $F_{\ell\lambda}(0)$ mass is $(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$. It is easily shown that the the arrival rate of *successful* draws is then $\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$. The distribution of waiting times until the first success is an exponential distribution with this parameter. The survivor function is therefore: $e^{-\bar{\lambda}(1-F_{\ell\lambda}(0)-F_{h\lambda}(0))t}$. Due to the total mass of one, $F_{\ell\lambda}(0)+F_{h\lambda}(0)) \in (0, 1)$, so the survivor function is decreasing in t. $F_{\ell\lambda}(0)$ is independent of the $\bar{\lambda}$ arrival rate when taking limits as no agents enter this region from successful searches. Taking the limit for any t, $\lim_{\bar{\lambda}\to\infty} e^{-\bar{\lambda}(1-F_{\ell\lambda}(0)-F_{h\lambda}(0))t} = 0$. Therefore, in the limit in equilibrium, $F_{\ell\lambda}(0) = 0$ as measure 1 agents get a successful draw in any strictly positive interval. The same arguments can be used to explain why $F_{h\lambda}(0) = 0$.

To ensure that the expected search costs in this limit are finite, calculate the present discounted value of flow payments until the first success. This is the exponential distribution with parameter $\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$ and flow cost $\zeta\bar{\lambda}$

$$\mathbb{E}\left[\text{search costs}\right] = \int_0^\infty \left(\int_0^t \zeta\bar{\lambda}e^{-rs}\mathrm{d}s\right)\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))e^{-\bar{\lambda}(1-F_{\ell\lambda}(0)-F_{h\lambda}(0))t}\mathrm{d}t \tag{C.1}$$

$$= \frac{\bar{\lambda}\zeta}{r + \bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))} \tag{C.2}$$

Taking the limit and use the result that $F_{\ell\lambda}(0)$ and $F_{h\lambda}(0) \to 0$

$$\lim_{\lambda\to\infty} \mathbb{E}\left[\text{search costs}\right] = \zeta \tag{C.3}$$

Therefore, in the limit the model can have draws directly from above the current threshold, with measure 0 remaining behind, and a cost for an instantaneous adoption of ζ.

C.2 Recall Doesn't Change the Optimal Stopping Conditions

As in McCall search, the ability to reject a draw and recall the last productivity will have no effect in equilibrium. To see this, assume firms can accept or reject a draw, but still draw from the truncated distribution above $M(t)$ for simplicity (see Appendix C.1 for why this isn't an important assumption). To show that this doesn't change the optimal policy, it is necessary to compare the optimal stopping conditions (i.e., value matching and smooth pasting) to show that they are identical with and without recall.

The value-matching condition is not effected by recall, as conditioning on a successful draw it is above $M(t)$. Hence accept or reject has no effect on the value matching condition for the stopping problem.

The smooth pasting condition is more complicated and needs to be analyzed assuming deviations from the $M(t)$ stopping rule. If an agent draws, then the net value they gain is the probability that they draw above Z and accept, plus the probability they draw below Z and keep Z. Conditional on acceptance, the distribution of $\tilde{Z}$ is the drawing distribution truncated above acceptance, i.e., the current Z. Writing out the gross value of adoption,

$$V_s(t,Z) = (1-\Phi(t,Z))\int_Z^\infty V(t,\tilde{Z})\frac{\Phi'(t,\tilde{Z})}{1-\Phi(t,Z)}\mathrm{d}\tilde{Z} + \Phi(t,Z)V(t,Z) \tag{C.4}$$

$$= \int_Z^\infty V(t,\tilde{Z})\Phi'(t,\tilde{Z})\mathrm{d}\tilde{Z} + \Phi(t,Z)V(t,Z) \tag{C.5}$$

Differentiate this with respect to Z and use the fundamental theorem of calculus,

$$\boldsymbol{\partial}_Z V_s(t,Z) = -V(t,Z)\Phi'(t,Z) + \Phi(t,Z)\boldsymbol{\partial}_Z V(t,Z) + \Phi'(t,Z)V(t,Z) \tag{C.6}$$

$$= \Phi(t,Z)\boldsymbol{\partial}_Z V(t,Z) \tag{C.7}$$

The smooth pasting condition is evaluated at $M(t)$, and $\Phi(t,M(t)) = 0$ at points of continuity,

$$\boldsymbol{\partial}_Z V_s(t,M(t)) = 0 \tag{C.8}$$

Therefore, the smooth pasting condition is identical to the original version without recall.

C.3 Independent Draw of Z and Type with Infinite Support

Write the adoption process more generally to allow some correlation between the draw of the type and productivity, for some constants a_ℓ, b_ℓ, a_h, b_h (constrained with some equilibrium conditions on the masses of $F_\ell(\infty)$ and $F_h(\infty)$,

$$0 = gF_\ell'(z) - \lambda_\ell F_\ell(z) + \lambda_h F_h(z) + a_\ell F_\ell(z) + b_\ell F_h(z) - S_\ell \tag{C.9}$$

$$0 = (g-\gamma)F_h'(z) - \lambda_h F_h(z) + \lambda_\ell F_\ell(z) + a_h F_\ell(z) + b_h F_h(z) - S_h \tag{C.10}$$

With this, the C matrix for the KFE in Main Paper (A.73) is,

$$C \equiv \begin{bmatrix} \frac{\lambda_\ell}{g} - \frac{a_\ell}{g} & -\frac{b_\ell}{g} - \frac{\lambda_h}{g} \\ -\frac{a_h}{g-\gamma} - \frac{\lambda_\ell}{g-\gamma} & \frac{\lambda_h}{g-\gamma} - \frac{b_h}{g-\gamma} \end{bmatrix} \tag{C.11}$$

The determinant of C is,

$$\det(C) = \frac{a_\ell(b_h - \lambda_h) - a_h(b_\ell + \lambda_h) - \lambda_\ell(b_h + b_\ell)}{g(g-\gamma)} \tag{C.12}$$

If $g > \gamma$: In that case, A necessary condition for the convergence of the KFE is that the eigenvalues of C are both negative. As the determinant is the product of the eigenvalues, the determinant must be positive. For this determinant, a necessary condition is therefore

$$a_\ell b_h > a_h(b_\ell + \lambda_h) + a_\ell \lambda_h + \lambda_\ell(b_h + b_\ell) \tag{C.13}$$

For independent draws of the type and productivity, for some θ, a, b, the parameters must be related through $a_\ell = \theta a, b_\ell = \theta b, a_h = (1-\theta)a, b_h = (1-\theta)b$. It can be shown that (C.13), is false for any a, b, θ. Therefore, the adoption technology must have correlation between the draw of i and the draw of Z.

If $g < \gamma$: Here, we do not have the same problem since the denominator of (C.12) is negative. In fact, these equilibria are possible with certain initial conditions and could be solved using a technique similar to that of $g < \gamma$ in Main Paper Section 3.2.1.

C.4 Unbounded Technology Frontier, No Bounded Equilibrium for any κ

Define the following to simplify notation,

$$\alpha \equiv (1+\hat{\lambda})\frac{S}{g} \tag{C.14}$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h} \tag{C.15}$$

$$\bar{\lambda} \equiv \frac{\lambda_\ell}{r - g + \lambda_h} + 1 \tag{C.16}$$

$$\nu = \frac{(r-g)\bar{\lambda}}{g} \tag{C.17}$$

$$F_h(z) = \hat{\lambda} F_\ell(z) \tag{C.18}$$

Substitute into Main Paper (48) to form a first-order non-linear ODE in $F_\ell(z)$

$$0 = S(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa + gF_\ell'(z) - S \tag{C.19}$$

Rearrange,

$$F_\ell'(z) = \tfrac{S}{g} - \tfrac{S}{g}(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa \tag{C.20}$$

This non-linear ODE is separable,

$$\mathrm{d}z = \frac{\mathrm{d}F_\ell(z)}{\frac{S}{g} - \frac{S}{g}(1+\hat{\lambda})^\kappa F_\ell(z)^\kappa} \tag{C.21}$$

Integrate,

$$z + C_1 = \int_0^{F_\ell} \frac{1}{\frac{S}{g} - \frac{S}{g}(1+\hat{\lambda})^\kappa q^\kappa} \mathrm{d}q \tag{C.22}$$

Define the following function of $q \in [0,1]$, where ${}_1\mathbb{F}_2(\cdot)$ is the Hypergeometric function.[11]

$$Q_\ell(q) = \frac{g}{S} q \; {}_1\mathbb{F}_2 \left(1, 1/\kappa, 1 + 1/\kappa, (1+\hat{\lambda})^\kappa q^\kappa\right) \tag{C.26}$$

Assume that $Q_\ell(q)$ has an inverse $Q_\ell^{-1}(\cdot)$. Use (C.23) and (C.22) and (C.26),

$$z + C_1 = Q_\ell(F_\ell(z)) \tag{C.27}$$

$$F_\ell(z) = Q_\ell^{-1}(z + C_1) \tag{C.28}$$

From (C.26) and Main Paper (23), $Q^{-1}(C_1) = 0$. From (C.26), $C_1 = Q_\ell(0) = 0$. Since $C_1 = 0$ and $F_\ell(z) = Q_\ell^{-1}(z)$, $Q_\ell(q)$ is the quantile function for the random variable z.

To show that there doesn't exist a bounded support solution, assume a $\bar{z} < \infty$. Use Main Paper (24) and (A.53) to get:

$$F_\ell(\bar{z}) = \tfrac{1}{1+\hat{\lambda}} \tag{C.29}$$

Use the definition of Q_ℓ

$$\bar{z} = Q_\ell^{-1}(\tfrac{1}{1+\hat{\lambda}}) \tag{C.30}$$

From (C.26)

$$\bar{z} = \tfrac{g}{(1+\hat{\lambda})S} \; {}_1\mathbb{F}_2 \left(1, 1/\kappa, 1 + 1/\kappa, 1\right) \tag{C.31}$$

But, for any parameters, the Hypergeometric function is only defined on $(-1, 1)$ and goes to infinity on the boundaries. Hence, $\bar{z} \to \infty$ and there can be no stationary equilibrium with bounded support.

Appendix D Robustness of Empirics and Calibration

This section collects robustness checks, including alternative parameterizations.

[11] One definition for the Hypergeometric function is derived by simplifying Euler's representation in its integral form, is,

$$\int_0^y \frac{1}{a + b\tilde{y}^\kappa} d\tilde{y} = \frac{y}{a} \; {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa, -\tfrac{b}{a} y^\kappa) \tag{C.23}$$

Derivatives of Hypergeometric functions can use the following result:

$$\tfrac{\mathrm{d}}{\mathrm{d}p} \, {}_1\mathbb{F}_2(a, b, c; p) = \frac{ab}{c} \; {}_1\mathbb{F}_2(a+1, b+1, c+1; p) \tag{C.24}$$

In our specific case, other transformations can be made,

$$\tfrac{\mathrm{d}}{\mathrm{d}p} \, {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa; p) = \frac{\frac{1}{1-p} - {}_1\mathbb{F}_2(1, 1/\kappa, 1 + 1/\kappa; p)}{\kappa p} \tag{C.25}$$

	(1)				
	mean	p25	p50	p75	sd
Log(p99/p5) Revenue	5.30	3.48	5.17	6.93	2.61
Log(p99/p5) Employment	4.98	3.31	5.03	6.65	2.35
% Change Log(p99/p5) Revenue	-1.85	-28.27	-0.18	26.90	75.95
% Change Log(p99/p5) Employment	0.77	-22.87	0.00	23.71	69.01
Observations	12818				

Table 1: Alternative Frontier Statistics (Using p99/p05 by SIC, 1980-2014)

D.1 Empirics of the Technology Frontier Robustness

To give a longer sample, Figure 2 provides a panel for the pooled industries, starting in 1961. The post-1980 was used due to small sample biases, and a general concern with data quality in compustat. For example, the average number of firms in the sample within each cell is 9.6 for years before 1980 and 16.3 for those above 1980. Biases, especially since percentiles are sensitive to small numbers of firms, would creep into the statistics.

Figure 2: Time series of the frontier (pooled industries, 1961-2014)

As a robustness check on the 90-10 percentile ratio of Main Paper Table 1, Table 1 provides the summary statistics on the growth rate and level of the 99-05 frontier proxy. The results remain qualitatively the same.

A summary of stationarity tests on the panel of industries, as shown in Main Paper Figure 19, is given in Table 2. The frontier dynamics on the SIC panel with the 90/10 ratio is done in Table 3 and Figure 3.

	(1)	
	mean	sd
KPSS accept stationarity of Revenue	0.75	0.43
Phillips-Perron reject unit-root of Revenue	0.23	0.42
Both accept stationarity of Revenue	0.20	0.40
One or more accept stationarity of Revenue	0.78	0.42
KPSS accept stationarity of Employment	0.75	0.43
Phillips-Perron reject unit-root of Employment	0.20	0.40
Both accept stationarity of Employment	0.18	0.38
One or more accept stationarity of Employment	0.77	0.42
Observations	11097	

Table 2: Summary of Stationarity Tests (by SIC, 1990-2014)

	(1)			
	mean	p50	p25	p75
Revenue Frontier Ratio (1991-2000)/(1981-1990)	1.09	0.98	0.84	1.20
Employment Frontier Ratio (1991-2000)/(1981-1990)	1.10	1.00	0.85	1.22
Revenue Frontier Ratio (2001-2010)/(1991-2000)	0.89	0.87	0.68	1.05
Employment Frontier Ratio (2001-2010)/(1991-2000)	0.92	0.93	0.72	1.08
Revenue Frontier Ratio (2001-2010)/(1981-1990)	0.94	0.86	0.65	1.10
Employment Frontier Ratio (2001-2010)/(1981-1990)	0.98	0.89	0.67	1.14
Observations	12763			

Table 3: Frontier Ratio Summary Statistics (Using p99/p05 by SIC)

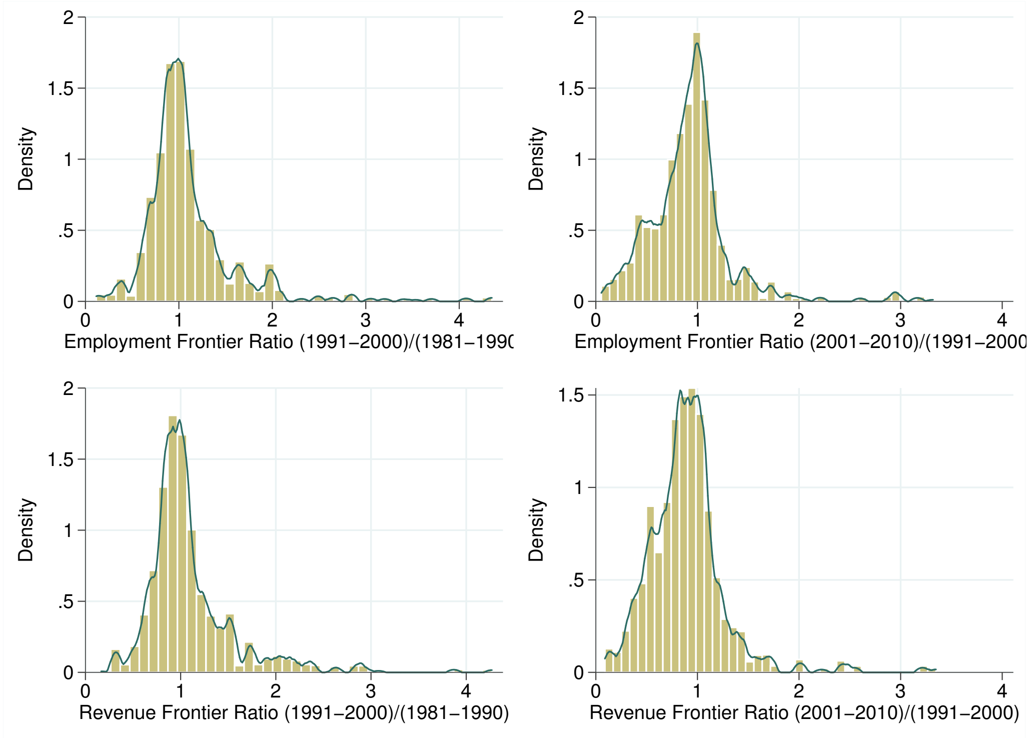

Figure 3: Histogram of Frontier Ratios (Using p99/p05 by SIC)

D.2 Alternative Parameterization with Bounded Support

As an example, use the standard parameter values, except calibrate $\eta = 0.0097$ and $\zeta = 17.8291$ to match $\alpha = 2.12$ and $\bar{z} = 0.651$ discussed in *Main* Appendix D.6.

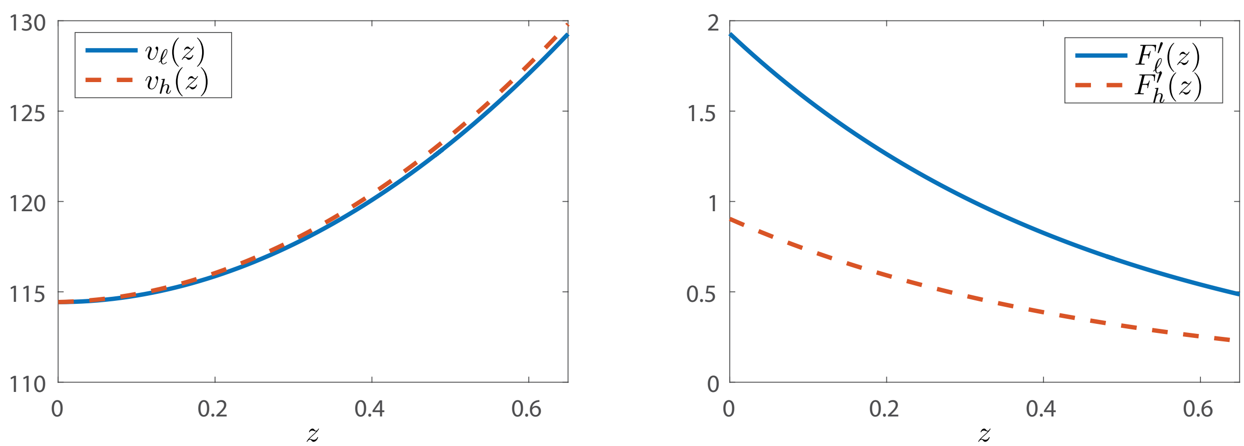

Figure 4: Exogenous $v_i(z)$, and $F_i'(z)$ with a Bounded Frontier with $\bar{z} = 0.651$ calibration

D.3 Alternative Parameterization of the Infinite Support

Curiously, with other parameters the growth rate can instead be increasing in α, that is as the tail gets thinner, as shown in Figure 6. The change in the monotonicity of $g(\alpha)$ Figure 6 is accompanied by a change in the monotonicity of $F_h(\infty)$. Recall that with the draw technology in Main Paper

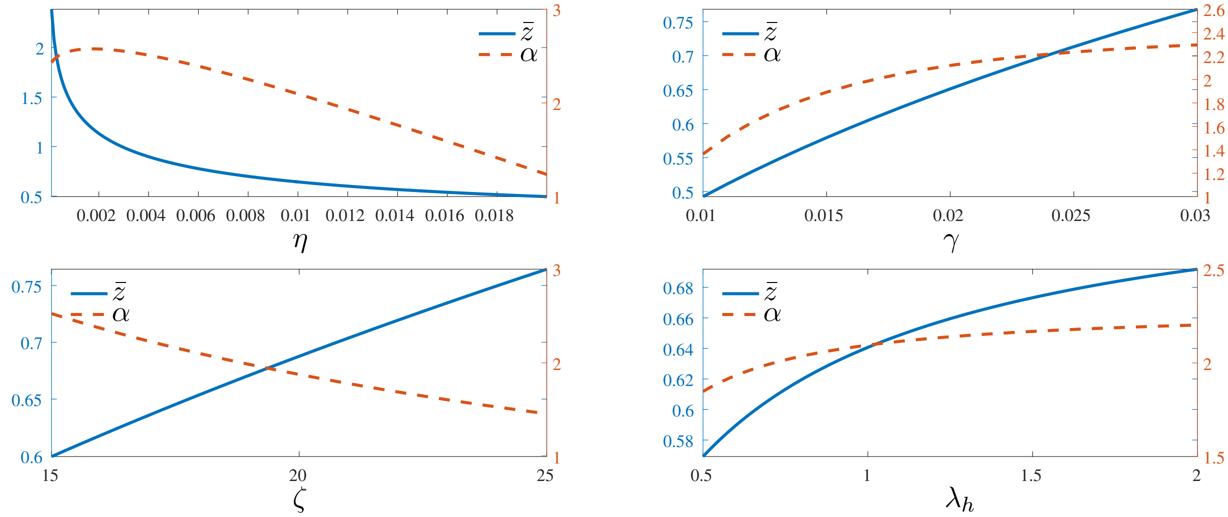

Figure 5: Comparative Statics with a Bounded Frontier with $\bar{z} = 0.651$ calibration

(32), where both the i and the z are imitated, an increasing proportion of ℓ types means more S_ℓ crossing the adoption boundary that can end up in the growth state h, generating a higher $g(\alpha)$.

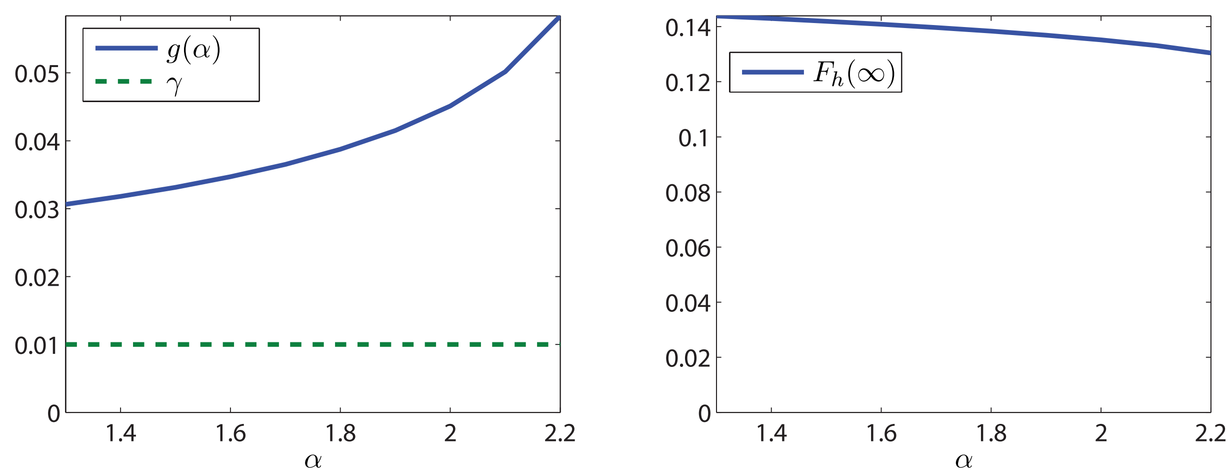

Figure 6: Growth rate as a Function of α with an Infinite Frontier with Alternative Parameters

Appendix E Innovation with Geometric Brownian Motion

This section summarizes propositions and results for a version of our model using Geometric Brownian Motion instead of the H/L Markov chain. The BGP with GBM is given in Appendix E.4, while that of deterministic innovation (i.e., a special case of GBM with no volatility) is given in Appendix E.5. In the deterministic setup, we go further and derive the transition dynamics from any initial condition in Appendix E.6.

E.1 Model Summary

We begin by describing features of the baseline model of diffusion in the absence of a finite technology frontier, similar to Perla, Tonetti, and Waugh (2015) and Perla and Tonetti (2014). To investigate the role of stochastic innovation, this model nests a stochastic, exogenous innovation process modeled as Geometric Brownian Motion (GBM).[12,13]

Comparing to Main Paper Section 2, firms are now only heterogeneous over their productivity, with CDF $\Phi(t, Z)$.

Diffusion and Evolution of the Distribution If a firm adopts a new technology, then it immediately changes its productivity to a draw from the distribution $\Phi(t, Z)$, potentially distorted. The degree of imperfect mobility is indexed by $\kappa > 0$ where the agent draws its Z from the CDF $\Phi(t, Z)^\kappa$. Note that for higher κ, the probability of a better draw increases. As $\Phi(t, \bar{Z}(t))^\kappa = 1$ and $\Phi(t, M(t))^\kappa = 0$, for all $\kappa > 0$, this is a valid probability distribution. As before, in equilibrium, all firms choose an identical threshold, $M(t)$, above which they will continue operating with their existing technology.

A flow $S(t) \geq 0$ of firms cross into the adoption region at time t and choose to adopt a new technology. For the case in which innovation is driven by GBM, with a drift of γ and variance σ, the Kolmogorov Forward Equation (KFE) below (in CDFs) is[14]

$$\partial_t \Phi(t, Z) = \underbrace{-(\gamma - \sigma^2/2) Z\, \partial_Z \Phi(t, Z)}_{\text{Deterministic Drift}} + \underbrace{\frac{\sigma^2}{2} Z^2\, \partial_{ZZ} \Phi(t, Z)}_{\text{Brownian Motion}} + \underbrace{S(t)\Phi(t, Z)^\kappa - S(t)}_{\text{Firm draws - Adopters}}, \quad \text{for } M(t) \leq Z \leq \bar{Z}(t) \tag{E.1}$$

If $\sigma > 0$, then $\bar{Z}(t) = \infty$ immediately. Otherwise, if $\sigma = 0$ and $\bar{Z}(t) < \infty$, then the frontier grows at rate $\bar{Z}'(t)/\bar{Z}(t) = \gamma$.

E.2 Firm's Problem

The firm maximizes the present discounted value of profits, discounting at rate $r > 0$, where Z evolves following a GBM. The firm chooses the productivity threshold $M(t)$, below which they choose to adopt a new technology. A firm's productivity may hit $M(t)$ due to a sequence of bad relative shocks or because the $M(t)$ barrier is overtaking their Z.[15]

Assuming continuity of $\Phi(0, Z)$, then the necessary conditions for an equilibrium, $\Phi(t, Z)$ and $M(t)$, are,

[12] Staley (2011) adds exogenous geometric Brownian motion to an economy with a Lucas (2009) technology diffusion model, and investigates the evolution of the productivity distribution and growth rates.

[13] For a version of the model using monopolistic competition and associated equilibrium conditions, see Appendix G.

[14] To derive from the more common KFE written in PDFs, use the adjoint of the infinitesimal generator of GBM,

$$\frac{\partial \phi(t, Z)}{\partial t} = -\frac{\partial}{\partial Z}\left((\mu + \upsilon^2/2) Z \phi(t, Z)\right) + \frac{\partial}{\partial Z^2}\left(\frac{\upsilon^2}{2} Z^2 \phi(t, Z)\right) + \ldots$$

Integrate this with respect to Z to convert into CDF $\Phi(t, Z)$, apply the final derivative to the 3rd term, and then rearrange to find,

$$\frac{\partial \Phi(t, Z)}{\partial t} = (\upsilon^2/2 - \mu) Z \frac{\partial \Phi(t, Z)}{\partial Z} + \frac{\upsilon^2}{2} Z^2 \frac{\partial^2 \Phi(t, Z)}{\partial Z^2} + \ldots$$

[15] The sequential formulation and connection to a recursive optimal stopping of a deterministic process is given on page 110-112 of Stokey (2009).

$$rV(t,Z) = Z + (\gamma + \sigma^2/2)Z\,\boldsymbol{\partial}_Z V(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}V(t,Z) + \boldsymbol{\partial}_t V(t,Z) \tag{E.2}$$

$$V(t,M(t)) = \int_{M(t)}^{\bar{Z}(t)} V(t,\hat{Z})\mathrm{d}\Phi(t,\hat{Z})^{\kappa} - \zeta M(t) \tag{E.3}$$

$$\boldsymbol{\partial}_Z V(t,M(t)) = 0, \quad \text{if } S(t) > 0 \tag{E.4}$$

$$\boldsymbol{\partial}_t \Phi(t,Z) = -(\gamma - \sigma^2/2)Z\,\boldsymbol{\partial}_Z\Phi(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}\Phi(t,Z) + S(t)\Phi(t,Z) - S(t) \tag{E.5}$$

$$\Phi(t,M(t)) = 0 \tag{E.6}$$

$$\Phi(t,\bar{Z}(t)) = 1 \tag{E.7}$$

$$\bar{Z}'(t)/\bar{Z}(t) = \gamma, \text{ if } \sigma > 0 \tag{E.8}$$

where equation (E.2) is the Bellman Equation in the continuation region, and equations (E.3) and (E.4) are the value-matching and smooth-pasting conditions. While the value-matching condition always holds, the smooth-pasting condition is only necessary if there is negative drift relative to the boundary $M(t)$. Equations (E.5) to (E.7) are the Kolmogorov forward equation with the appropriate boundary conditions. Equation (E.8) is the deterministic growth of the boundary, which is simply the growth rate of frontier agents as $M(t) < \bar{Z}(t)$ in equilibrium.

E.3 Normalization and Stationarity

Following the normalization of Appendix F.1, leads to the following normalized set of equations. Given an initial condition $F(0,z)$, the dynamics of $v(t,z)$, $F(t,z)$, $g(t) \geq 0$, and $S(t) \geq 0$, must satisfy

$$(r - g(t))v(t,z) = e^z + (\gamma - g(t))\boldsymbol{\partial}_z v(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz}v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{E.9}$$

$$v(t,0) = \int_0^{\infty} v(t,z)\mathrm{d}F(t,z)^{\kappa} - \zeta \tag{E.10}$$

$$\boldsymbol{\partial}_z v(t,0) = 0 \tag{E.11}$$

$$0 = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz}F(t,z) + S(t)F(t,z)^{\kappa} - S(t) \tag{E.12}$$

$$F(t,0) = 0 \tag{E.13}$$

$$F(t,\bar{Z}(t)) = 1 \tag{E.14}$$

$$S(t) = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,0) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz}F(t,0) \tag{E.15}$$

In stationary form, these become $v(z)$, $F(z)$, $g \geq 0$, $S > 0$, and $0 < \bar{z} \leq \infty$ such that,

$$(r - g)v(z) = e^z + (\gamma - g)v'(z) + \tfrac{\sigma^2}{2}v''(z) \tag{E.16}$$

$$v(0) = \int_0^{\infty} v(z)\mathrm{d}F(z)^{\kappa} - \zeta \tag{E.17}$$

$$v'(0) = 0 \tag{E.18}$$

$$0 = (g - \gamma)F'(z) + \tfrac{\sigma^2}{2}F''(z) + SF(z) - S \tag{E.19}$$

$$F(0) = 0 \tag{E.20}$$

$$F(\infty) = 1 \tag{E.21}$$

$$S = (g - \gamma)F'(0) + \tfrac{\sigma^2}{2}F''(0) \tag{E.22}$$

The value-matching condition in (E.17) can also be written, using Main Paper (B.16), as

$$\zeta = \int_0^{\infty} v'(z)\left(1 - F(z)^{\kappa}\right)\mathrm{d}z \tag{E.23}$$

To interpret (E.23), in equilibrium, as the firm is already about to gain $v(0)$ costlessly, it is indifferent between production and adoption only if the sum of all marginal values over the counter-CDF of draws is identical to the cost of adoption. (E.22) can be understood as the flux crossing the endogenous barrier, where in normalized terms the barrier is moving at rate $g-\gamma$ and collecting the infinitesimal mass at the boundary, i.e., the PDF $F'(0)$. Additionally, there is a Brownian diffusion term where a σ dependent flow of agents are moving back purely randomly.

E.4 Stochastic Exogenous Innovation

If innovation is stochastic and driven by GBM, even with a finite $F(0, z)$ initial condition, the support of a stationary $F(z)$ must be $[0, \infty)$. With a continuum of agents, Brownian motion instantaneously increases the support of the distribution.

When geometric random shocks are added, the stationary solutions will endogenously become power-law distributions, as discussed with generality in Gabaix (2009). Main Paper Figure 3 provides some intuition on how these forces can create a stationary distribution with technology diffusion. Stochastic innovation spreads out the distribution and in the absence of endogenous adoption this would prevent the existence of a stationary distribution.[16] However, as the distribution spreads, the incentives to adopt a new technology increase, and this in turn acts to compress the distribution. In equilibrium, technology diffusion occurs with certainty because otherwise the returns to adopt a new technology become infinite in relative terms.

To see this intuitively, consider the alternative where there are geometric stochastic shocks for operating firms, but no firm chooses to adopt new technologies. A distribution generated by a random walk has a growing variance, and its support is unbounded unless there is adoption or death, even if the drift is zero. But when firms are choosing whether to adopt a new technology or not, the increasing variance of the distribution implies the returns to adoption go to infinity, overcoming any finite adoption cost. The firms at the lower end of the productivity distribution would choose to adopt, and the spread of the distribution would be contained.

Proposition 1 (Equilibrium with Geometric Brownian Motion Innovations)**.** *A continuum of equilibria parameterized by α exist satisfying*

$$\alpha > \frac{1}{2}\left(1+\sqrt{\frac{4+\zeta(r-\gamma-\sigma^2/2)}{\zeta(r-\gamma-\sigma^2/2)}}\right) \tag{E.24}$$

and

$$0 > (\alpha-1)^2\alpha\zeta^2\sigma^4 - 2(\alpha-1)\zeta\sigma^2\left((\alpha-3)\alpha+\left(\alpha^2-1\right)\zeta(r-\gamma)\right)+4((\alpha-1)\zeta(r-\gamma)-1)^2. \tag{E.25}$$

For a given α, the growth rate is

$$g = \underbrace{\gamma}_{Innovation} + \underbrace{\frac{1-(\alpha-1)\zeta\,(r-\gamma)}{(\alpha-1)^2\zeta}}_{Deterministic\ Latent\ Growth} + \underbrace{\frac{\sigma^2}{2}\frac{\alpha\left(\alpha(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-2\right)+1}{(\alpha-1)\left((\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-1\right)}}_{Stochastic\ Latent\ Growth}. \tag{E.26}$$

Furthermore the stationary distribution and value function are

$$F(z) = 1-e^{-\alpha z} \tag{E.27}$$

$$v(z) = \frac{1}{r-\gamma-\sigma^2/2}\left(e^z+\frac{1}{\nu}e^{-\nu z}\right), \tag{E.28}$$

[16] Without endogenous adoption there is no “absorbing" or “reflecting" barrier and geometric random shocks lead to a diverging variance in the KFE.

where,

$$\nu = \frac{\gamma - g}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}}. \tag{E.29}$$

Proof. See Appendix F.2. □

The continuum of equilibria in this case is similar to that discussed in Luttmer (2007, 2012, 2015). While there exist multiple stationary equilibria, the uniqueness of a stationary equilibrium given a particular initial condition in a similar model is discussed in Luttmer (2015). This corresponds to hysteresis (i.e., dependence on initial condition $\Phi(0, Z)$): a unique path exists given a particular set of parameters and initial conditions. Furthermore, given α, the flow of adopters, S, satisfies

$$\alpha = \frac{g-\gamma}{\sigma^2} - \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 - \frac{S}{\sigma^2/2}}, \tag{E.30}$$

that is,

$$S = \frac{\sigma^2}{2}\left(\alpha - \frac{g-\gamma}{\sigma^2}\right)^2 + \left(\frac{g-\gamma}{\sigma^2}\right)^2. \tag{E.31}$$

The tail index in (E.30) is of a very similar form to that in Luttmer (2015) Proposition 1, where our endogenous flow of adopters, S, is related to Luttmer's exogenous arrival rate of learning opportunities.

The growth rate and value function as $\sigma \to 0$ is identical to that in Proposition 2. Hence, in the decomposition of growth rates of (E.26), the first term is the latent growth caused by the same incentives as in the deterministic case, while the second is some latent growth caused by negative (unlucky) shocks to firms close to the adoption threshold. While some firms will receive positive shocks and be lucky when near the threshold, half of the drift-adjusted Brownian motion ends up crossing the threshold.

Another place where the Brownian motion can be seen to influence the equilibrium is in (E.29). This is a stochastic version of the ν in (E.35). Higher variance decreases the expected hitting time at which productivity reaches the normalized zero threshold, and hence increases the value of technology diffusion. Note that, due to risk neutrality, the variance is constructed to have no direct effect on the expected continuation profits, just on the expected length of time the firm will operate its existing technology.

To decompose the contributions to growth, define the growth rate with no stochastic shocks to productivity as $g_c(\alpha)$ as in (E.32). Since the contributions to growth rates from the stochastic part of latent growth (i.e., unlucky experimentation pushing firms below the boundary in relative terms, as interacted with initial conditions) have been removed, this can be interpreted as the deterministic element of latent growth.

$$g_c(\alpha) \equiv \lim_{\sigma \to 0} g(\alpha; \sigma) \tag{E.32}$$

Figure 7 shows a plot of $g(\alpha)$ from equation (E.26) and $g_c(\alpha)$ from (E.32), for parameter values: $r = 0.06, \zeta = 25, \sigma = .1$, and $\gamma = 0$. The range of admissible α from the intersection of the sets in (E.24) and (E.25) is relatively tight, between about 1.49 and 1.62. As discussed before, a stationary equilibrium with strictly positive Brownian term and no equilibrium technology diffusion is not possible, so the $\alpha \simeq 1.62$ is likely the stationary distribution for a large number of initial

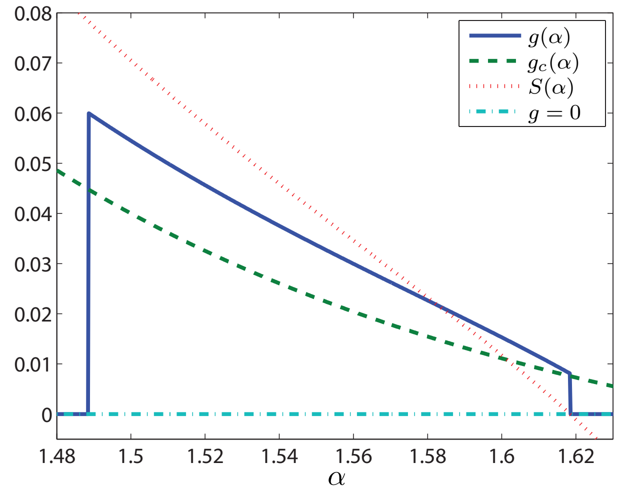

Figure 7: Growth rate and Growth from Catchup Diffusion as a Function of α

conditions with relatively thin tails. Equilibria where $\alpha < 1.49$ here are simply not defined as the option value explodes.[17]

The minimum growth rate of around 1% occurs at the maximum α, and is strictly positive. This occurs at the point where the contribution of stochastic part of latent growth, $g - g_c$, is zero. Otherwise, the contribution is strictly positive in the admissible range

E.5 Deterministic Balanced Growth Path

Here we analyze the deterministic balanced growth path, in which $\sigma = 0$, and innovation is common and constant for all firms. First we assume that firms are adopting technologies from the unconditional distribution by setting $\kappa = 1$. Proposition 2 characterizes the BGP equilibrium, which is the continuous time version of Perla and Tonetti (2014) with a Pareto initial condition.

Proposition 2 (Deterministic Equilibrium with Pareto Initial Condition and $\kappa = 1$)**.** *If* $\Phi(0, Z) = 1 - \left(\frac{M_0}{Z}\right)^{\alpha}$, $\alpha > 1$, *and* $r > \gamma + (\zeta\alpha(\alpha - 1))^{-1} > 0$, *then*

$$g = \frac{1 - \zeta(\alpha - 1)(r - \alpha\gamma)}{\zeta(\alpha - 1)^2}, \tag{E.33}$$

$$v(z) = \frac{1}{r - \gamma} e^{z} + \frac{1}{\nu(r - \gamma)} e^{-\nu z}, \tag{E.34}$$

where,

$$\nu \equiv \frac{r - g}{g - \gamma} > 0, \tag{E.35}$$

and the stationary distribution in logs is

$$F(z) = 1 - e^{-\alpha z}. \tag{E.36}$$

Proof. See Appendix F.3. □

[17] In determining whether these α are empirically plausible, consider the crude adjustment to tail indices based on for markups discussed in (G.30).

The first term of equation (E.34) is the value of production in perpetuity. This would be the value of the firm if it did not have the option of adopting a better technology. The second term of equation (E.34) is the *option value of technology diffusion*. It is decreasing in z since the optimal time to adopt is increasing with better relative technologies. The exponent, ν in (E.35) determines the rate at which the option value is discounted. More discounting of the future, or slower growth rates, lead to a more rapid drop-off of this option value.

For $\kappa > 0$, the draws are distorted and the stationary distribution is a non-Pareto power-law.

Definition 1 (Power Law Distribution)**.** *A distribution $\Phi(Z)$ is defined as a power-law, or equivalently is fat-tailed, if there exists an $\alpha > 0$ such that for large Z, the counter-CDF is asymptotically Pareto $1 - \Phi(Z) \approx Z^{-\alpha}$. Under the change of variables $z \equiv \log(Z)$ with $F(z) \equiv \Phi(e^z)$, the distribution $\Phi(Z)$ is a power-law if the counter-CDF is asymptotically exponential $1 - F(z) \approx e^{-\alpha z}$.*

Pareto distributions trivially fulfill the requirements of a Power Law. See Appendix F.5.1 for more formal definition based on the theory of regularly varying functions, and Appendix F.4 for more on the tail index α. We say a distribution is thin-tailed if there does not exist any $\alpha > 0$ such that the definition can hold.

The following proposition generalizes Proposition 2 for $\kappa > 0$.

Proposition 3 (Deterministic Equilibrium with Pareto Initial Condition and $\kappa > 0$)**.** *For a given g, the value function is independent of κ. For $\nu \equiv \frac{r-g}{g-\gamma}$,*

$$v(z) = \frac{1}{r-\gamma}e^z + \frac{1}{\nu(r-\gamma)}e^{-\nu z} \tag{E.37}$$

Given $F'(0)$ parameterizing the set of stationary equilibria, the quantile of $F(z)$ is defined as $Q(q) = F^{-1}(q)$, as a form based on the Hypergeometric function, ${}_1\mathbb{F}_2(\cdot)$.

$$Q(q) = \frac{q}{F'(0)}\ {}_1\mathbb{F}_2\left(1, 1/\kappa, 1 + 1/\kappa, q^\kappa\right) \tag{E.38}$$

The power law tail index of $F(z)$ is

$$\alpha = \kappa F'(0) \tag{E.39}$$

Given a $F'(0)$, the equilibrium growth rate fulfills the following equation

$$\frac{1}{r-g} + \zeta = \int_0^1 \left(\frac{1}{r-\gamma}e^{Q(q^{1/k})} + \frac{1}{\nu(r-\gamma)}e^{-\frac{r-g}{g-\gamma}Q(q^{1/k})}\right) \mathrm{d}q \tag{E.40}$$

Proof. See Appendix F.4. □

The definition in (E.38) provides an expression for the quantile of the distribution, and (E.40) can be solved numerically for g. In the simple case of a Pareto distribution and $\kappa = 1$, $\alpha = F'(0)$, so (E.39) nests the Pareto case of $\kappa = 1$. Intuitively, as κ grows, the draws are more skewed towards the lower tail and hence the thinner tailed distribution. As the $\kappa > 0$ case features the same qualitative behavior as the $\kappa = 1$ case, but is less analytically tractable, we will concentrate on $\kappa = 1$ for most of our analysis.

E.6 Deterministic Transition Dynamics

In this section, we characterize the transition dynamics of an economy that features deterministic innovation, i.e., $\sigma = 0$. For the dynamics, it will sometimes be more convenient to calculate the growth rate of M at the state M. This can be translated to t from the law of motion if $g(t) > 0$ on some interval $t \in [0, T]$ for $T \leq \infty$.[18] Denote $\hat{g}(M) = M'(M)/M$. On a balanced growth path, $g = M'(t)/M$, which also is the growth rate of output. For notational simplicity, define a truncation of the initial condition, $\Phi(0, Z)$, at M as,

$$\Phi_M(Z) \equiv \frac{\Phi(0,Z) - \Phi(0,M)}{1 - \Phi(0,M)}, \quad Z \in [M, \infty) \tag{E.41}$$

with PDF

$$\Phi'_M(Z) \equiv \frac{\Phi'(0,Z)}{1 - \Phi(0,M)}, \quad Z \in [M, \infty). \tag{E.42}$$

For simplicity, set $\kappa = 1$ and $\gamma = 0$ to analyze the de-trended version of the model with unconditional draws. To simplify algebra, let the cost of adoption be ζZ rather than $\zeta M(t)$.[19] Under these conditions, Proposition 4 describes the growth rate along the transition path and its dependence on initial conditions.

Proposition 4 (Dynamic Solution for an Arbitrary Initial Condition)**.** *The growth rate of M given an initial condition $\Phi(0, Z)$ is determined by is given by*

$$\hat{g}(M) = \frac{\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1 + \zeta r)}{\zeta M \Phi'_M(M)}. \tag{E.43}$$

Additionally,

1. *On a BGP, if $\phi(0, Z)$ thin-tailed then $\lim\limits_{M\to\infty} \hat{g}(M) = \gamma$. The asymptotic distribution does not converge to the frontier, and is not degenerate, so $\lim_{t\to\infty} M(t)/\bar{Z}(t) < 1$.*
2. *Otherwise, on a BGP with $g > \gamma$, then $\Phi(0, Z)$ has power law tails and the asymptotic growth rate is determined by the tail parameter α of its distribution, where*[20]

$$g = \gamma + \frac{1 - \zeta(r - \gamma)(\alpha - 1)}{\zeta\alpha(\alpha - 1)} \tag{E.44}$$

Proof. See Appendix F.5.1. (E.43) can be extended to the $\gamma > 0$ case by simply adding the γ trend to the growth of M. □

The key result is that growth driven through technology diffusion can only continue forever if the distribution has power law tails. Figure 8 shows an example of a Frechet initial condition converging towards the growth rate determined from its power law tail. The initial growth rate is higher since the non-monotonicity of the Frechet density provides strong incentives for catching up, and the tail becomes relatively thinner as z increases.

[18] To see this, note that if $g(t) > 0$ on some interval, then $M(t)$ is strictly increasing. Therefore, due to the lack of aggregate shocks, $M(t)$ is bijective on this interval.

[19] In the recursive formulation with a ζZ cost, the only change to the set of equations is that the smooth pasting condition becomes $\boldsymbol{\partial}_Z V(t, Z) = -\zeta$. This ends up simplifying the algebra for solving for the transition dynamics. In the rest of the paper, we use the cost function $\zeta M(t)$. Recall that the gross value of search is independent of Z. Hence, an economic reason to use a non-Z dependent cost in the other sections is if we believe the cost should be symmetrically independent of Z. See Appendix F.5.1 for details.

[20] The difference between (E.44) and the $\sigma = 0$ case of (E.26) come from the alternative cost function.

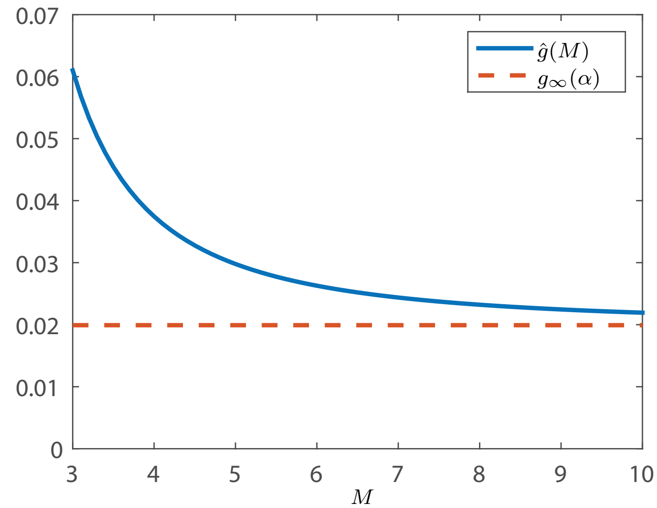

Figure 8: Dynamics from a Frechet Initial Condition

Without an initial power-law tail, the growth rate is asymptotically determined by the innovation rate. In that case, growth from catching up stops when the productivity distribution compresses enough to balance the returns from innovation with the returns from adoption at the margin: the adoption threshold grows at the same rate as that of innovation–i.e., no long-run latent growth. Hence, no firms will hit the diffusion threshold as all are growing with deterministic innovation.[21]

Appendix F Detailed Proofs for Geometric Brownian Motion

This section provides detailed proofs on the model with GBM described in Appendix E.

F.1 Normalization and Stationarity

As we no longer have an i index, and the KFEs and Bellman equations now have a second-order term, the normalization is different from that of Main Paper Appendix A.1.

Normalizing the Productivity Distribution Define the normalized distribution of productivity, as the distribution of productivity relative to the endogenous adoption threshold $M(t)$:

$$\Phi(t, Z) \equiv F(t, \log(Z/M(t))) \tag{F.1}$$

Differentiate to obtain the PDF

$$\boldsymbol{\partial}_Z \Phi(t, Z) = \frac{1}{Z} \frac{\partial F(t, \log(Z/M(t)))}{\partial z} = \frac{1}{Z} \boldsymbol{\partial}_z F(t, z) \tag{F.2}$$

[21]In order to see if an endogenous choice of R&D intensity changes these results within the deterministic version of the model, we can add an innovation decision for operating firms within this simple framework. As it can be shown that the innovation rate is weakly increasing in z, it is conceivable that in a stationary equilibrium, agents near the adoption barrier may choose not to innovation, which could lead to equilibrium technology diffusion. However, as in the exogenous innovation case, with a finite productivity distribution, the only stationary equilibrium is one in which every firm does (or does not) conduct innovation and there is no equilibrium technology diffusion. Endogenous innovation within the stochastic model is discussed in Main Paper Section 4.

Differentiate again,

$$\boldsymbol{\partial}_{ZZ}\Phi(t,Z) = \frac{1}{Z^2}\left(\boldsymbol{\partial}_{zz}F(t,z) - \boldsymbol{\partial}_z F(t,z)\right) \tag{F.3}$$

Differentiate (F.1) with respect to t and use the chain rule to obtain the transformation of the time derivative

$$\boldsymbol{\partial}_t\Phi(t,Z) = \frac{\partial F(t,\log(Z/M(t)))}{\partial t} - \frac{M'(t)}{M(t)}\frac{\partial F(t,\log(Z/M(t)))}{\partial z} \tag{F.4}$$

Use the definition $g(t) \equiv M'(t)/M(t)$ and the definition of z,

$$\boldsymbol{\partial}_t\Phi(t,Z) = \boldsymbol{\partial}_t F(t,z) - g(t)\boldsymbol{\partial}_z F(t,z) \tag{F.5}$$

Normalizing the Law of Motion Substitute (F.2), (F.3) and (F.5) into (E.5),

$$\begin{aligned}\frac{\partial F(t,\log(Z/M(t)))}{\partial t} - g(t)\frac{\partial F(t,\log(Z/M(t)))}{\partial z} &= -(\gamma - \sigma^2/2)\frac{\partial F(t,\log(Z/M(t)))}{\partial z} \\ &+ \frac{\sigma^2}{2}\left(\frac{\partial^2 F(t,\log(Z/M(t)))}{\partial z^2} - \frac{\partial F(t,\log(Z/M(t)))}{\partial z}\right) \\ &+ S(t)F(t,\log(Z/M(t)))^\kappa - S(t)\end{aligned} \tag{F.6}$$

Use the definition of z and reorganize to find the normalized KFE,

$$\boldsymbol{\partial}_t F(t,z) = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz}F(t,z) + S(t)F(t,z)^\kappa - S(t) \tag{F.7}$$

For $S(t)$, evaluate (F.7) at $z = 0$

$$S(t) = (g(t) - \gamma)\boldsymbol{\partial}_z F(t,0) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz}F(t,0) \tag{F.8}$$

Normalizing the Value Function Define the normalized value of the firm as,

$$v(t,\log(Z/M(t))) \equiv \frac{V(t,Z)}{M(t)} \tag{F.9}$$

Rearrange,

$$V(t,Z) = M(t)v(t,\log(Z/M(t))) \tag{F.10}$$

Differentiate (F.10) with respect to t

$$\boldsymbol{\partial}_t V(t,Z) = M'(t)v(t,\log(Z/M(t))) - M'(t)\frac{\partial v(t,\log(Z/M(t))}{\partial z} + M(t)\frac{\partial v(t,\log(Z/M(t))}{\partial t} \tag{F.11}$$

Divide by $M(t)$ and use the definition of $g(t)$

$$\frac{1}{M(t)}\boldsymbol{\partial}_t V(t,Z) = g(t)v(t,z) - g(t)\boldsymbol{\partial}_z v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{F.12}$$

Differentiate (F.10) with respect to Z,

$$\boldsymbol{\partial}_Z V(t,Z) = \frac{M(t)}{Z}\frac{\partial v(t,\log(Z/M(t)))}{\partial z} \tag{F.13}$$

$$\boldsymbol{\partial}_{ZZ} V(t,Z) = \frac{M(t)}{Z^2}\left(-\frac{\partial v(t,\log(Z/M(t)))}{\partial z} + \frac{\partial^2 v(t,\log(Z/M(t)))}{\partial z^2}\right) \tag{F.14}$$

Rearrange,

$$\frac{1}{M(t)}\boldsymbol{\partial}_Z V(t,Z) = \frac{1}{Z}\boldsymbol{\partial}_z v(t,z) \tag{F.15}$$

$$\frac{1}{M(t)}\boldsymbol{\partial}_{ZZ} V(t,Z) = \frac{1}{Z^2}\left(\boldsymbol{\partial}_{zz} v(t,z) - \boldsymbol{\partial}_z v(t,z)\right) \tag{F.16}$$

Divide (E.2) by $M(t)$ and then substitute from (F.12), (F.15) and (F.16),

$$\begin{aligned} r\frac{1}{M(t)}V(t,Z) = {} & \frac{Z}{M(t)} + (\gamma+\sigma^2/2)\frac{M(t)}{M(t)}\frac{Z}{Z}\boldsymbol{\partial}_z v(t,z) + \frac{\sigma^2}{2}\left(\boldsymbol{\partial}_{zz} v(t,z)) - \boldsymbol{\partial}_z v(t,z)\right) \\ & + g(t)v(t,z) - g(t)\boldsymbol{\partial}_z v(t,z) + \boldsymbol{\partial}_t v(t,z) \end{aligned} \tag{F.17}$$

Use (F.10) and the definition of z and rearrange (F.17),

$$(r-g(t))v(t,z) = e^z + (\gamma - g(t))\boldsymbol{\partial}_z v(t,z) + \tfrac{\sigma^2}{2}\boldsymbol{\partial}_{zz} v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{F.18}$$

Optimal Stopping Conditions Divide the value matching condition in (E.3) by $M(t)$,

$$\frac{V(t,M(t))}{M(t)} = \int_{M(t)}^{\bar{Z}(t)} \frac{V(t,Z)}{M(t)}\boldsymbol{\partial}_Z\Phi(t,Z)\Phi(t,Z)^{\kappa-1}\mathrm{d}Z - \frac{M(t)}{M(t)}\zeta \tag{F.19}$$

Use the substitutions in (F.2) and (F.9)

$$v(t,0) = \int_{M(t)}^{\bar{Z}(t)} v(t,\log(Z/M(t)))\frac{1}{Z}\frac{\partial F(t,\log(Z/M(t)))}{\partial z}F(t,\log(Z/M(t)))^{\kappa-1}\mathrm{d}Z - \zeta \tag{F.20}$$

Use the change of variable $z = \log(Z/M(t))$ in the integral, which implies $\mathrm{d}z = \frac{1}{Z}\mathrm{d}Z$. Note that the bounds of integration change to $[\log(M(t)/M(t)), \log(\bar{Z}(t)/M(t))] = [0, \bar{z}(t)]$

$$v(t,0) = \int_0^{\bar{z}(t)} v(t,z)\boldsymbol{\partial}_z F(t,z)F(t,z)^{\kappa-1}\mathrm{d}z - \zeta \tag{F.21}$$

Which can also be written succinctly as,

$$v(t,0) = \int_0^{\bar{z}(t)} v(t,z)\mathrm{d}F(t,z)^{\kappa} - \zeta \tag{F.22}$$

Evaluate (F.15) at $Z = M(t)$ to find that

$$\frac{\partial V(t,M(t))}{\partial Z} = \boldsymbol{\partial}_z v(t,0) \tag{F.23}$$

Substitute this into (E.4) to give the smooth pasting condition as,

$$\boldsymbol{\partial}_t v(t,0) = 0 \tag{F.24}$$

As a model variation, if the cost is proportional to Z, then the only change to the above conditions is that the smooth pasting condition becomes $\boldsymbol{\partial}_z v(t,0) = -\zeta$. This cost formulation has the potentially unappealing feature that the value is not monotone in Z, as firms close to the adoption threshold would rather have a lower Z to decrease the adoption cost for the same benefit.

F.2 Derivation of Stationary Equilibrium

Proof. Guess that the form of the solution of (E.19) and (E.20) is,

$$F(z) = 1 - e^{-\alpha z} \tag{F.25}$$

The boundary value (E.20) is fulfilled. Substitute this guess into (E.19) and collect terms to use undetermined coefficients to find

$$0 = S + \alpha(\gamma - g + \alpha\sigma^2/2) \tag{F.26}$$

Solve this equation and choose the non-explosive root,

$$\alpha = \frac{g-\gamma}{\sigma^2} - \sqrt{\frac{(g-\gamma)^2}{\sigma^4} - \frac{S}{\sigma^2/2}} \tag{F.27}$$

To solve (E.16) and (E.18), assume a solution of the form,

$$v(z) = ae^z + \frac{b}{\nu}e^{-\nu z} \tag{F.28}$$

Substitute this guess into (E.16) and (E.18) and equate undetermined coefficients to find the following system of equations

$$0 = -a\gamma + ar - \frac{a\sigma^2}{2} - 1 \tag{F.29}$$

$$0 = b\gamma - \frac{bg}{\nu} - bg - \frac{1}{2}b\nu\sigma^2 + \frac{br}{\nu} \tag{F.30}$$

$$0 = a - b \tag{F.31}$$

The absence of z confirms that a solution to this system is a particular solution of the ODE and initial condition. While there could be another exponential term for this 2nd order equation, use a no-bubble condition to eliminate it, and choose the positive ν root in the quadratic. The solution to this system of equations is,

$$a = \frac{1}{r - \gamma - \sigma^2/2} \tag{F.32}$$

$$\nu = \frac{\gamma - g}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}} \tag{F.33}$$

$$b = a \tag{F.34}$$

And the substituted solution is,

$$v(z) = \frac{1}{r - \gamma - \sigma^2/2}\left(e^z + \frac{1}{\nu}e^{-\nu z}\right) \tag{F.35}$$

Use (F.35) to find,

$$v(0) = \frac{\frac{1}{\nu} + 1}{r - \gamma - \frac{\sigma^2}{2}} \tag{F.36}$$

Substitute (F.25), (F.35) and (F.36) into (E.17) to get an equation relating α and ν,

$$\frac{\frac{1}{\nu}+1}{r-\gamma-\frac{\sigma^2}{2}} = \int_0^\infty \left(\frac{\alpha}{r-\gamma-\sigma^2/2} e^{-(\alpha-1)z} + \frac{\alpha}{\nu(r-\gamma-\sigma^2/2)} e^{-(\alpha+\nu)z}\right) \mathrm{d}z - \zeta \tag{F.37}$$

$$= \frac{\alpha(\nu+1)(\alpha+\nu-1)}{(\alpha-1)\nu(\alpha+\nu)(r-\gamma-\sigma^2/2)} - \zeta \tag{F.38}$$

We assumed that $\alpha > 1$ and will analyze interior cases where $\nu > 0$ and the integral converges. Simplify further,

$$0 = (\alpha-1)(\alpha+\nu)\zeta + \frac{\nu+1}{\gamma-r+\sigma^2/2} \tag{F.39}$$

This can be solved further for ν,

$$\nu = \frac{1-(\alpha-1)\alpha\zeta(r-\gamma-\sigma^2/2)}{(\alpha-1)\zeta(r-\gamma-\sigma^2/2)-1} = \frac{\alpha-1}{1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta} - \alpha \tag{F.40}$$

Equate (F.33) and (F.40) to find an expression in g and α

$$0 = \gamma - g + \alpha\sigma^2 + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} + \frac{(\alpha-1)\sigma^2}{(\alpha-1)(r-\gamma-\sigma^2/2)\zeta-1} \tag{F.41}$$

Solve for g,

$$g = \frac{-\left((\alpha-1)\alpha\zeta\left(2\gamma+\sigma^2\right)+2\right)\left((\alpha-1)\zeta\left(\alpha\sigma^2+2\gamma\right)+2\right)-4(\alpha-1)^2\zeta^2r^2+2(\alpha-1)\zeta r\left((\alpha-1)\zeta\left(\left(\alpha^2+1\right)\sigma^2+2(\alpha+1)\gamma\right)+4\right)}{2(\alpha-1)^2\zeta\left((\alpha-1)\zeta\left(-2\gamma+2r-\sigma^2\right)-2\right)} \tag{F.42}$$

Decompose,

$$g = \frac{1-(\alpha-1)\zeta\,(r-\alpha\gamma)}{(\alpha-1)^2\zeta} + \frac{\sigma^2}{2}\,\frac{\alpha\left(\alpha(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-2\right)+1}{(\alpha-1)\left((\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta-1\right)} \tag{F.43}$$

Define the contibution to growth from deterministic latent growth in the decomposition of (F.43) as,

$$g_c = \frac{1-(\alpha-1)\zeta\,(r-\alpha\gamma)}{(\alpha-1)^2\zeta} \tag{F.44}$$

From (F.40), a necessary condition for (F.43) to be the positive solution to (F.41) is,

$$0 < 1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta \tag{F.45}$$

Furthermore, assuming (F.45) and manipulating (F.40) gives an upper bound,

$$\frac{\alpha-1}{\alpha} > 1-(\alpha-1)\left(r-\gamma-\frac{\sigma^2}{2}\right)\zeta \tag{F.46}$$

Reorganize (F.45) and (F.46), solve the quadratic, and choose the positive root to get necessary—but not sufficient—bounds on α,

$$\frac{1}{2}\left(1+\sqrt{\frac{4+\zeta(r-\gamma-\sigma^2/2)}{\zeta(r-\gamma-\sigma^2/2)}}\right) < \alpha < 1 + \frac{1}{\zeta(r-\gamma-\sigma^2/2)} \tag{F.47}$$

Upper Bound on α Take (F.27) and solve for S,

$$S = \alpha\left(g - \gamma - \alpha\frac{\sigma^2}{2}\right) \tag{F.48}$$

The condition for positive S is therefore,

$$g > \gamma + \alpha\sigma^2/2 \tag{F.49}$$

To find a bound, substitute (F.43) into (F.49),

$$\begin{aligned} 0 < &\left((\alpha-1)^2\alpha\zeta^2\sigma^4 - 2(\alpha-1)\zeta\sigma^2\left((\alpha-3)\alpha + \left(\alpha^2-1\right)\zeta(r-\gamma)\right) + 4((\alpha-1)\zeta(r-\gamma)-1)^2\right) \\ &\times\left((\alpha-1)\zeta\left(2\gamma - 2r + \sigma^2\right) + 2\right) \end{aligned} \tag{F.50}$$

Note that from the assumptions in (F.47), the 2nd term is negative, and for negative g from (F.50) it is necessary and sufficient for the first term to be negative.

$$0 > (\alpha-1)^2\alpha\zeta^2\sigma^4 - 2(\alpha-1)\zeta\sigma^2\left((\alpha-3)\alpha + \left(\alpha^2-1\right)\zeta(r-\gamma)\right) + 4((\alpha-1)\zeta(r-\gamma)-1)^2 \tag{F.51}$$

This is a cubic equation, solving for α here gives a complicated expression for a maximum α such that $S > 0$. A lower bound on α is defined by (E.24), while the upper bound on α is a root of the cubic (E.25) at equality. □

F.3 BGP of Model with Deterministic, Exogenous Innovation

This section is effectively the continuous time version of stationary solutions in Perla and Tonetti (2014).

Proof of Proposition 2. Solve (E.19) with $\sigma = 0$ and with the initial condition (E.20),

$$F(z) = 1 - e^{-\frac{S}{g-\gamma}z} \tag{F.52}$$

From (E.22), given a $F'(0)$,

$$F(z) = 1 - e^{-F'(0)z} = 1 - e^{-\alpha z} \tag{F.53}$$

From the normalization of the initial condition $\Phi(0, Z)$, it can be shown that it is already in the form of (F.53), and $F'(0) = \alpha$ at time 0, so the stationary distribution remains on a balanced growth path for the $\alpha \equiv F'(0)$ in the initial condition.

To solve (E.16) and (E.18), assume a solution of the form,

$$v(z) = ae^z + be^{-\nu z} \tag{F.54}$$

Substitute this guess into (E.16) and (E.18) and equate undetermined coefficients to find the following system of equations

$$0 = -a\gamma + ar - 1 \tag{F.55}$$

$$0 = b\gamma\nu - bg\nu - b + br \tag{F.56}$$

$$0 = a - b\nu \tag{F.57}$$

Solve this system, and recognize that the guess of (F.54) is confirmed and there is no z in the system,

$$\nu = \frac{r-g}{g-\gamma} \tag{F.58}$$
$$a = \frac{1}{r-\gamma} \tag{F.59}$$
$$b = \frac{a}{\nu} \tag{F.60}$$

Substitute these back into the guess to get the value function

$$v(z) = \frac{1}{r-\gamma}e^{z} + \frac{1}{\nu(r-\gamma)}e^{-\nu z} \tag{F.61}$$

Substitute (F.53) and (F.61) into (E.17), and integrate and simplify

$$1 = (\alpha-1)\zeta(-\alpha\gamma + (\alpha-1)g + r) \tag{F.62}$$

Solve for g,

$$g = \frac{1-\zeta(\alpha-1)(r-\alpha\gamma)}{\zeta(\alpha-1)^2} \tag{F.63}$$

To derive the parameter restriction on r, note that $r > g > \gamma$ is necessary for a non-explosive (F.61). Substitute for g and simplify,

$$r > \gamma + \frac{1}{\zeta\alpha(\alpha-1)} \tag{F.64}$$

□

F.4 BGP of Model Without Stochastic Innovation and $\kappa > 0$

Proof of Proposition 3. The solution techniques for $v(z)$ are identical to that in Appendix F.3. However, the KFE is now in general a nonlinear ODE. Rearrange (E.19),

$$F'(z) = \tfrac{S}{g-\gamma} - \tfrac{S}{g-\gamma}F(z)^{\kappa} \tag{F.65}$$

This non-linear ODE is separable,

$$\mathrm{d}z = \frac{\mathrm{d}F(z)}{\frac{S}{g-\gamma} - \frac{S}{g-\gamma}F(z)^{\kappa}} \tag{F.66}$$

Integrate,

$$z + C_1 = \int_0^F \frac{1}{\frac{S}{g-\gamma} - \frac{S}{g-\gamma}q^{\kappa}}\mathrm{d}q \tag{F.67}$$

Define ${}_1\mathbb{F}_2$ as the Hypergeometric function according to (C.23). Define the following function of $q \in [0,1]$,

$$Q(q) = \frac{g-\gamma}{S} q \; {}_1\mathbb{F}_2\left(1, 1/\kappa, 1+1/\kappa, q^{\kappa}\right) \tag{F.68}$$

Assume that $Q(q)$ has an inverse $Q^{-1}(\cdot)$. Use (C.22), (C.23) and (F.68),

$$z + C_1 = Q(F(z)) \tag{F.69}$$

$$F(z) = Q^{-1}(z + C_1) \tag{F.70}$$

From (E.21) and (F.68), $Q^{-1}(C_1) = 0$. From (F.68), $C_1 = Q(0) = 0$. Since $C_1 = 0$ and $F(z) = Q^{-1}(z)$, $Q(q)$ is the quantile function for the random variable z. From (E.22), we can get the stationary distribution indexed by a $F'(0)$ as in the example with $\kappa = 1$). Writing the stationary quantile function,

$$Q(q) = \frac{q}{F'(0)}\ {}_1\mathbb{F}_2\left(1, 1/\kappa, 1 + 1/\kappa, q^\kappa\right) \tag{F.71}$$

From probability theory, using the quantile function of the log of a random variable, the tail index is,[22]

$$\alpha = \lim_{q\to 1} \frac{Q''(q)}{(Q'(q))^2} \tag{F.72}$$

Use (C.25)

$$= \lim_{q\to 1} \kappa F'(0) q^{\kappa-1}, \tag{F.73}$$

which shows the distortion of the tail parameter by κ relative to the Exponential distribution with $\kappa = 1$

$$\alpha = \kappa F'(0) \tag{F.74}$$

This nests the results of a standard Pareto, $\alpha = F'(0)$.

To calculate the value-matching condition, an expectation must be taken over $\hat{F}(z) \equiv F(z)^\kappa$. The quantile function solves the equation $F(z) = q$ for z. For the new quantile function solving $\hat{F}(z) = \hat{q}$, use the definition, $\hat{F}(z) = F^\kappa(z) = \hat{q}$, and transform to find $F(z) = \hat{q}^{1/\kappa}$. Therefore, the quantile for $\hat{F}$ is $\hat{Q}(q) = Q(q^{1/\kappa})$

Recall that expectations can be written in quantile functions, i.e., given a CDF $F(z)$ and quantile $F^{-1}(p)$, then for $h(\cdot)$,

$$E(h(X)) = \int_{-\infty}^{\infty} h(x)F'(x)dx = \int_0^1 h(F^{-1}(p))dp \tag{F.75}$$

Using this, integrate over $\hat{F}(z)$ with the quantile $\hat{Q}(q)$ in (E.17)

$$\frac{1}{r-g} + \zeta = \int_0^1 v(\hat{Q}(q))\mathrm{d}q \tag{F.76}$$

Substitute for $v(z)$ and $\hat{Q}(\cdot)$ in this function to get an implicit equation in g □

F.5 Dynamic Solution of the Deterministic Model

This section solves for the full dynamics, maintaining that $\kappa = 1$.

[22] See http://stats.stackexchange.com/questions/104169/limiting-expression-for-power-law-tail-index-from-a-quantile-function for a derivation

De-Trending The γ trend can be removed through a change of variables. Let $\tilde{Z} = e^{\gamma t} Z$, $\tilde{V}(t, e^{-\gamma t} Z) = e^{-\gamma t} V(t, Z)$, and $\tilde{\Phi}(t, e^{-\gamma Z} Z) = \Phi(t, Z)$. Differentiate these, and substitute into the Bellman Equation in (E.16),

$$\tilde{r}\tilde{V}(t, \tilde{Z}) = \tilde{Z} + \frac{\partial \tilde{V}(t, \tilde{Z})}{\partial t} \tag{F.77}$$

Where $\tilde{r} = r - \gamma$. Substitute into (E.19),

$$\frac{\partial \tilde{\Phi}(t, \tilde{Z})}{\partial t} = S(t)\tilde{\Phi}(t, \tilde{Z})^{\kappa} - S(t) \tag{F.78}$$

Similarly $\tilde{M}(t) \equiv e^{-\gamma t} M(t)$, $M'(t)/M(t) \equiv \gamma + \tilde{M}'(t)/\tilde{M}(t)$, and $\tilde{\bar{Z}}(t) \equiv e^{-\gamma t} \bar{Z}(t)$ This provides a system of equations in the same form as that without drift. The remainder of this section as well as Appendices F.3, F.5 and F.5.1 solve the model with $\gamma = 0$ drift for notational simplicity to conserve on the tildes, with a transformation done at the end.

General Solution to the Law of Motion in Deterministic Model The general solution will be proved by guess-and-verify

For any initial condition with CDF and PDF $\Phi_0(Z)$, $\Phi_0'(Z)$ guess that the PDF of the solution is

$$\boldsymbol{\partial}_Z \Phi(t, Z) = \frac{\Phi_0'(Z)}{1 - \Phi_0(M(t))} \tag{F.79}$$

If we take this equation and plug it into (E.5), we see that

$$\frac{\Phi_0'(Z)\phi(M(t))M'(t)}{(1 - \Phi_0(M(t)))^2} = \frac{\Phi_0'(Z)}{1 - \Phi_0(M(t))} \frac{\Phi_0'(M(t))}{1 - \Phi_0(M(t))} M'(t) \tag{F.80}$$

Recalling the truncation notation introduced in (E.41) and (E.42), this means the general solution is

$$\Phi_{M(t)}'(Z) \equiv \frac{\Phi_0'(Z)}{1 - \Phi_0(M(t))} \tag{F.81}$$

which holds for any $M(t)$ and $\Phi_0(Z)$.

Alternative Cost Functions and the Sequential Formulation Given the de-trending above, assume $\gamma = 0$ and drop the tildes.

Given a $V_s(t)$ *gross* value of search at time t, define $T(Z, t)$ as the (relative) time to search. This is related to the optimal adoption threshold through $M(t) \equiv \max\{Z | T(Z, t) = 0\}$. In the de-trended setup, a firm operates until it adopts a new technology at calendar time $t + T$.

With a cost function of ζZ, the sequential formulation is

$$V(t, Z) = \max_{T \geq 0} \left\{ \int_0^T e^{-r\tau} Z \mathrm{d}\tau + e^{-rT} \left[V_s(t + T) - \zeta Z\right] \right\} \tag{F.82}$$

$$= \max_{T \geq 0} \left\{ \tfrac{1 - e^{-rT}}{r} Z + e^{-rT} \left[V_s(t + T) - \zeta Z\right] \right\} \tag{F.83}$$

Simplification to ODE and Simple Integral Equation For algebraic simplicity, this will solve the version with the cost function ζZ instead of $\zeta M(t)$.

Because the problem is non-stochastic, it is possible to reduce it to a system of ODEs. Taking the first order condition of (F.83) for T.

$$0 = e^{-rT}\left(Z + r\zeta Z - rV_s(t+T) + V_s'(t+T)\right) \tag{F.84}$$

Hence, given $V_s(t)$, the solution to this ODE is a waiting time $T(t, Z)$. Focus on the the indifference point where $T = 0$, i.e., $M(t)$, Since the ODE in (F.85) needs to hold for all Z, evaluate it at $M(t)$ which has $T = 0$ by definition

$$rV_s(t) = (1 + r\zeta)M(t) + V_s'(t) \tag{F.85}$$

To get another equation from (F.83), we will need to find a way to eliminate $V(t, Z)$. Given an Z and the law of motion $M(t)$, we know that the time when $Z = M(t)$ is $M^{-1}(Z)$. By definition, T is the amount of time the agent will wait until they search. But given a solution $M(t)$, this means an agent with $Z > M(t)$ will wait for $T(t, Z) = M^{-1}(Z) - t$ until they search. As this T is the optimal choice, so we can plug it into (F.83) and drop the max

$$V(t, Z) = \tfrac{1-e^{-r(M^{-1}(Z)-t)}}{r}Z + e^{-r(M^{-1}(Z)-t)}\left[V_s(M^{-1}(Z)) - \zeta Z)\right] \tag{F.86}$$

Substitute (F.86) into (E.3), to get

$$V_s(t) = \int_{M(t)}^{B}\left(\tfrac{1}{r}Z - \tfrac{1+\zeta r}{r}e^{-r(M^{-1}(Z)-t)}Z + e^{-r(M^{-1}(Z)-t)}V_s(M^{-1}(Z))\right)\mathrm{d}\Phi_{M(t)}(Z) \tag{F.87}$$

Hence the system has been reduced to two ordinary integro-differential equations, (F.85) and (F.87), in $M(t)$ and $V_s(t)$.

F.5.1 Dynamic Equilibrium with Deterministic Innovation

Proof of Proposition 4. The solution will take an arbitrary $\Phi_0(Z)$. Recall that, in general, $M(0) \neq \inf \text{support}\{\Phi_0\}$.

Transforming the Aggregate State M is a more natural aggregate state that is easier to normalize than t, and is bijective for strictly increasing economies.

Define M as the current threshold from $M(t)$ and assume invertibility due to strictly increasing $M(t)$ (at least until growth stops).

$$t = M^{-1}(M) \equiv q(M) \tag{F.88}$$

Here $q(M)$ is just for notational convenience so we don't need to carry around the inverse. Use the inverse function theorem,

$$M'(t) = \frac{1}{q'(M(t))} = \frac{1}{q'(M)} \tag{F.89}$$

And use the definition of the growth rate,

$$g(t) \equiv M'(t)/M(t) = \frac{1}{q'(M(t))M(t)} \tag{F.90}$$

Finally, defining the post-change-of-variable growth rate of the adoption threshold as a function of the current threshold

$$g(q(M)) \equiv \hat{g}(M) = \frac{1}{Mq'(M)} \tag{F.91}$$

Use this q, we can transform the other functions. The derivatives are done using the chain rules.

$$H(M) \equiv V_s(q(M))e^{-rq(M)} \tag{F.92}$$

Differentiate and reorganizing,

$$H'(M) = e^{-rq(M)}q'(M)(V_s'(q(M)) - rV_s(q(M))) \tag{F.93}$$

$$V_s'(q(M)) = \frac{H'(M)e^{rq(M)}}{q'(M)} + rV_s(q(M)) \tag{F.94}$$

Substitute (F.92) and (F.94) into (F.87)

$$rV_s(q(M)) = (1+\zeta r)M + \frac{e^{rq(M)}}{q'(M)}H'(M) + rV_s(q(M)) \tag{F.95}$$

$$H'(M) = -(1+\zeta r)e^{-rq(M)}Mq'(M) \tag{F.96}$$

While the ODE is nonlinear, note that the $H(M)$ term has been removed. Take (F.83) and put in the change of variable,

$$V_s(q(M)) = \int_M^B \left(\tfrac{1-e^{-r(q(Z)-q(M))}}{r}Z - \zeta e^{-r(q(Z)-q(M))}Z + e^{-r(q(Z)-q(M))}V_s(q(Z))\right) \mathrm{d}\Phi_M(Z) \tag{F.97}$$

Rearrange (F.97)

$$(1-\Phi_0(M))V_s(q(M))e^{-rq(M)} = \int_M^B \left(\tfrac{1}{r}e^{-rq(M)}Z - \tfrac{(1+\zeta r)}{r}e^{-rq(Z)}Z + e^{-rq(Z)}V_s(q(Z))\right) \mathrm{d}\Phi_0(Z) \tag{F.98}$$

Use the $H(\cdot)$ definition with (F.98)

$$(1-\Phi_0(M))H(M) = \int_M^B \left(\tfrac{1}{r}e^{-rq(M)}Z - \tfrac{(1+\zeta r)}{r}e^{-rq(Z)}Z + H(Z)\right) \mathrm{d}\Phi_0(Z) \tag{F.99}$$

Expanding

$$\begin{aligned}(1-\Phi_0(M))H(M) = {} & \tfrac{1}{r}e^{-rq(M)}\int_M^\infty Z\mathrm{d}\Phi_0(Z) \\ & - \tfrac{(1+\zeta r)}{r}\int_M^\infty e^{-rq(Z)}Z\mathrm{d}\Phi_0(Z) \\ & + \int_M^B H(Z)\mathrm{d}\Phi_0(Z)\end{aligned} \tag{F.100}$$

Differentiate (F.100) with respect to M and use the rules of differentiation under the integral sign.

$$H'(M) = e^{-rq(M)}\left(\zeta M\Phi_M'(M+) - q'(M)\int_M^\infty Z\Phi_M'(Z)\mathrm{d}Z\right) \tag{F.101}$$

Substitute from (F.96). Note that the $e^{-rq(M)}$ drops out.

$$-(1+\zeta r)Mq'(M) = \zeta M\Phi'_M(M+) - q'(M)\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z \tag{F.102}$$

Simplify and solve for $q'(M)$

$$q'(M) = \frac{\zeta M\Phi'_M(M+)}{\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)M} \tag{F.103}$$

Or, writing as the growth rate of the threshold

$$\hat{g}(M) = \frac{\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)M}{\zeta M^2\Phi'_M(M+)} \tag{F.104}$$

$$= \frac{\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z - (1+\zeta r)}{\zeta M\Phi'_M(M+)} \tag{F.105}$$

Similar algebra can be done for the $\zeta M(t)$ cost function, but the final ODE in (F.103) ends up complicated and nonlinear.[23]

Thin and Fat Tailed Distributions A simple definition of a power law, or fat tailed, distribution is that in the limit, the PDF or counter-CDF is proportional to a power law for some tail index α. i.e.,

$$\lim_{x\to\infty}(1-F(x)) \propto x^{-\alpha} \tag{F.114}$$

$$\lim_{x\to\infty}F'(x) \propto x^{-1-\alpha} \tag{F.115}$$

[23]To sketch the algebra for the $\zeta M(t)$ cost function, the sequential form is

$$V(t,Z) = \max_{T\geq 0}\left\{\int_0^T e^{-r\tau}Z\mathrm{d}\tau + e^{rT}\left[V_s(t+T) - \zeta Z\right]\right\} \tag{F.106}$$

$$= \max_{T\geq 0}\left\{\tfrac{1-e^{-rT}}{r}Z + e^{-rT}\left[V_s(t+T) - \zeta M(t+T)\right]\right\} \tag{F.107}$$

If $\zeta M(t)$ is used instead, the equivalent to (F.85) is

$$rV_s(t) = (1-r\zeta)M(t) - \zeta M'(t) + V_s'(t) \tag{F.108}$$

If $\zeta M(t)$ is used instead, the equivalent to (F.87) is

$$V_s(t) = \int_{M(t)}^B \left(\tfrac{1}{r}Z - \tfrac{1}{r}e^{-r(M^{-1}(Z)-t)}Z + e^{-r(M^{-1}(Z)-t)}V_s(M^{-1}(Z)) - e^{-r(M^{-1}(Z)-t)}\zeta M(t)\right)\mathrm{d}\Phi_{M(t)}(Z) \tag{F.109}$$

Use (F.89), with the alternative ODE in (F.108), the transformed ODE,

$$H'(M) = e^{-rq(M)}\left(\zeta - (1+\zeta r)Mq'(M)\right) \tag{F.110}$$

Use the alternative cost function,

$$V_s(q(M)) = \int_M^B \left(\tfrac{1}{r}Z - \tfrac{1}{r}e^{-r(q(Z)-q(M))}Z + e^{-r(q(Z)-q(M))}V_s(q(Z)) - e^{-r(q(Z)-q(M))}\zeta M\right)\mathrm{d}\Phi_M(Z) \tag{F.111}$$

More formally, fat tails are defined as follows. Let a measurable function R defined on $(0, \infty)$ be *regularly varying with tail index* $\alpha \in (0, \infty)$ if

$$\lim_{x \to \infty} \frac{R\,(tx)}{R\,(x)} = t^{-\alpha}, \quad \forall t > 0$$

For $t > 1$, it is slowly varying if $\alpha = 0$ and rapidly varying if $\alpha = \infty$. (See Resnick (2007), p.13, 16). Then, a differentiable cumulative distribution function (CDF) $F(x)$ with counter-CDF $1 - F(x)$ is a power-law with tail index α if $1 - F(x)$ is regularly varying with index $\alpha > 0$. In this case we say the distribution is fat-tailed or has a power-law tail with index α. A standard example is the Pareto distribution, $F\,(x) = 1 - \left(\frac{x_m}{x}\right)^a$ defined over $[x_m, \infty)$ for x_m, $\alpha > 0$. A distribution with a power-law tail has integer moments equal to the highest integer below α ($\alpha = \infty$ means the distribution has all moments). The Cauchy distribution has a tail index of 1 and has no mean or higher moments. We may define a distribution that has all its moments (e. g the normal, lognormal distributions or distributions with bounded support) as thin-tailed. Obviously, the larger α, the "thinner" is the tail.
Define a thin-tailed distributions as those where this limit diverges and $\alpha = \infty$.

Fat-tailed and Perpetual Growth $\implies$ BGP from Tail Index: First, note that if growth continues forever, then $M \to \infty$, so assume that growth continues forever and the initial distribution is fat-tailed. Use the simple definition, if $\Phi(0, Z)$ is fat tailed, then $\Phi'(0, Z) \propto Z^{-1-\alpha}$ for large Z and some α. Use (E.42) and fix M,

$$\Phi'_M(Z) \propto \Phi'(0, Z), \quad \text{for } Z > M \tag{F.116}$$

As the fixed M goes to ∞, the domain of Z goes to ∞, and from (F.115),

$$\lim_{M \to \infty} \Phi'_M(Z) \propto Z^{-1-\alpha}, \quad \text{for } Z > M \tag{F.117}$$

Hence, for any fat-tailed initial condition, the asymptotic distribution under perpetual growth is a Pareto, and we can use our solution for a Pareto distribution based on the tail index.

Not Fat-tailed $\implies$ No Growth from Diffusion or Non-Existence: Denote the non-constant part of the denominator of the limit of (E.43) as $\hat{\alpha}$, and expand using (E.42),

$$\hat{\alpha} \equiv \lim_{M \to \infty} \frac{M\Phi'(M)}{1 - \Phi(M)} \leq \infty \tag{F.118}$$

With the alternative specification (F.113)

$$\begin{aligned}(1 - \Phi_0(M))H(M) = {} & \tfrac{1}{r} e^{-rq(M)} \int_M^\infty Z \mathrm{d}\Phi_0(Z) \\ & - \tfrac{1}{r} \int_M^\infty e^{-rq(Z)} Z \mathrm{d}\Phi_0(Z) \\ & - \zeta M \int_M^\infty e^{-rq(Z)} \mathrm{d}\Phi_0(Z) \\ & + \int_M^B H(Z) \mathrm{d}\Phi_0(Z) \end{aligned} \tag{F.112}$$

$$(1 - \Phi_0(M))V_s(q(M))e^{-rq(M)} = \int_M^B \left(\tfrac{1}{r} e^{-rq(M)} Z - \tfrac{1}{r} e^{-rq(Z)} Z + e^{-rq(Z)} V_s(q(Z)) - e^{-rq(Z)} \zeta M\right) \mathrm{d}\Phi_0(Z) \tag{F.113}$$

Compare (F.118) with the definition of regular variation to see that $\hat{\alpha}$ is finite if and only if the counter-CDF is regularly varying. For regularly varying functions, $\hat{\alpha} = \alpha$, the tail index of the distribution.

For distributions that are rapidly varying (i.e, not regularly or slowly varying), the denominator of (E.43) diverges to infinity.[24] As the numerator is bounded due to the assumption that expectations exist, this means that the asymptotic growth rate is $\hat{g} = 0$.

Recall that for an increasing, differentiable function $h(z)$ and a differentiable CDF $F(z)$, $z \in [0, \infty)$ (Note: if the lower support of z is $m > 0$, define $F'(z) = 0$ for $0 \le z < m$):

$$\mathbb{E}\left[h(z)\right] = \int h(z)F'(z)\mathrm{d}z = \int h'(z)(1 - F(z))\mathrm{d}z + h(\min \mathrm{support}\left\{F\right\}) \tag{F.119}$$

To show the numerator is bounded, reorganize (E.41)

$$1 - \Phi_M(Z) = \frac{1 - \Phi(0, Z)}{1 - \Phi(0, M)} \tag{F.120}$$

Then from (E.43), (F.119) and (F.120)

$$\frac{1}{M}\mathbb{E}\left[\Phi_M(z)\right] = \frac{1}{M}\int_M^\infty Z\Phi'_M(Z)\mathrm{d}Z \tag{F.121}$$

$$= \frac{1}{M}\frac{\int_M^\infty (1 - \Phi(0, Z))\mathrm{d}z}{1 - \Phi(0, M)} \tag{F.122}$$

Since $1 - \Phi(0, Z + M)$ is rapidly varying, there exists some M such that it is bounded by a regularly varying function for some $\alpha > 1$, i.e, $1 - \Phi(0, Z + M) < (Z + M)^{-\alpha}$ for all $Z > 0$. As $\frac{1}{M}\int_M^\infty Z\Phi'_M(Z)dZ$ can be calculated for this power-law (with a positive lower support), the numerator is bounded:

$$\frac{1}{M}\frac{\int_0^\infty (1 - \Phi(0, Z + M))dZ}{1 - \Phi(0, M)} = \frac{1}{M}\frac{\int_M^\infty (1 - \Phi(0, Z))dZ}{1 - \Phi(0, M)} = \frac{1}{M}\int_M^\infty Z\Phi'_M(Z)dZ < \frac{\alpha}{\alpha - 1}$$

Also note the case of the Cauchy where $1 - F(z)$ is slowly varying and $\hat{\alpha} = 0$. In that case, the denominator is 0 and the asymptotic growth rate is diverges and is not-defined. This could also be shown since the expectation of the the numerator of (E.43) is not defined in the case of the Cauchy. □

Appendix G Monopolistic Competition and Free Entry

This section adds monopolistic competition to the GBM in Appendix E. The core result is Proposition 5.

G.1 Unnormalized Model

The following sets up Geometric Brownian Motion version of the model, with monopolistic competition and technology diffusion costs in labor. The results are qualitatively similar to the simpler specification of the model. For simplicity, this section is written primarily for the balanced growth path. For exposition, drop the t subscript where possible.

Following a closed economy version of Perla, Tonetti, and Waugh (2015), the setup includes the standard setup of representative consumer purchasing a composite good with wages from inelastic labor supply and dividends from corporate profits. The composite good is produced by a competitive sector from a continuum of differentiated intermediates. A monopolistic firm, differentiated by its productivity Z, produces a single intermediate. Unlike in the previous examples, this will allow for an endogenous number of varieties determined by a free entry condition.

[24]See Resnick (2007) page 53 and 67 for a related characterization of rapid variation.

Consumer The consumer gains flow utility from consumption of final goods, $\frac{1}{1-\Lambda}C^{1-\Lambda}$, for $\Lambda \geq 0$. Future utility is discounted at a rate $\rho > 0$. The consumer purchases the final consumption goods purchased by supplying 1 unit of labor inelastically at real wage $W(t)$ and gaining profits from a perfectively diversified portfolio $\bar{\Pi}(t)$ in real profits. The consumer's welfare at time $\tilde{t}$ is then,

$$U(\tilde{t}) = \int_{\tilde{t}}^{\infty} \tfrac{C(t)^{1-\Lambda}}{1-\Lambda} e^{-\rho(t-\tilde{t})} \mathrm{d}t$$
$$\text{s.t.}\, P(t)C(t) = P(t)W(t) + P(t)\bar{\Pi}(t) \tag{G.1}$$

From standard asset pricing, the elasticity of inter-temporal substitution implies the interest rate used by the firm to discount future earnings,

$$\rho + \Lambda \tfrac{C'(t)}{C(t)} \tag{G.2}$$

On a balanced growth path, this gives the familiar interest rate with CRRA preferences. To find the firm's discounting rate, adjust it for an exogenous death rate of firms, $\delta \geq 0$

$$r \equiv \rho + \Lambda g + \delta \tag{G.3}$$

Final Goods Following the standard results from monopolistic competition, a competitive final goods sector produces a good with elasticity of substitution $\varpi > 1$ between all available products, given prices and final goods revenue denoted by PC. Let there be $N(t)$ varieties, and $\Phi(t, Z)$ the distribution of productivities for these varieties, normalized so that $\Phi(t, \infty) = 1$. Drop the t subscript for simplicity.

The solution follows from maximizing the following final goods production function,

$$\max_{Q} \left[\int_{M}^{\infty} Q(Z)^{(\varpi-1)/\varpi} \underbrace{N \mathrm{d}\Phi(Z)}_{\text{Unnormalized CDF}} \right]^{\varpi/(\varpi-1)} \tag{G.4}$$

$$\text{s.t.} \int_{M}^{\infty} p(Z)Q(Z)N\mathrm{d}\Phi(Z) = PC \tag{G.5}$$

Where $Q(Z)$ is the demand for a product with productivity Z. Defining a price index P, standard CES algebra gives the optimal intermediate good demand, price index, and final goods production as,

$$Q(Z) = \left(\tfrac{p(Z)}{P}\right)^{-\varpi} C \tag{G.6}$$

$$P = N^{\frac{1}{1-\varpi}} \left[\int_{M}^{\infty} p(Z)^{1-\varpi} \mathrm{d}\Phi(Z) \right]^{\frac{1}{1-\varpi}} \tag{G.7}$$

$$C = N^{\frac{\varpi}{1-\varpi}} \left[\int_{M}^{\infty} Q(Z)^{(\varpi-1)/\varpi} \mathrm{d}\Phi(Z) \right]^{\frac{\varpi}{1-\varpi}} \tag{G.8}$$

Static Variable Profits A monopolist, subject to the demand function in (G.6), maximizes real profits $\Pi(Z)$ by choosing the price $p(Z)$ and labor demand $\ell(Z)$

$$P\Pi(Z) \equiv \max_{p,\ell} \{\, (pZ\ell - PW\ell)\} \text{ s.t. (G.6)} \tag{G.9}$$

Define the markup $\hat{\varpi} \equiv \varpi/(\varpi - 1)$. The optimal solution to (G.9) gives $p(Z), \ell(Z)$, and $\Pi(Z)$,

$$\tfrac{p(Z)}{P} = \hat{\varpi} \tfrac{W}{Z} \tag{G.10}$$

$$\ell(Z) = \tfrac{Q(Z)}{Z} \tag{G.11}$$

Take (G.9) and divide by P to get $\Pi(Z) = \frac{p(Z)}{P}Q(Z) - \frac{Q(Z)}{Z}W$. Substitute from (G.10) as $W = \frac{Zp(Z)}{\bar{\varpi}P}$ to get $\Pi(Z) = \frac{1}{\varpi}\frac{p(Z)}{P}Q(Z)$. Finally, use $Q(Z)$ from (G.6) to get real profits

$$\Pi(Z) = \frac{1}{\varpi}\left(\frac{p(Z)}{P}\right)^{1-\varpi} C \tag{G.12}$$

Endogenous Varieties Assume that firm's exit at some exogenous $\delta \geq 0$ rate. To create a new variety, the entrepreneur must hire θ units of labor at the market wage, W, where $\theta > \zeta$. A new firm will adopt a random Z, with the same procedure as adopting firms. A free entry condition will determine the equilibrium N.

On a balanced growth path, a total of $\delta N \geq 0$ firms will both die and be created each period. This will use a total of $\delta\theta N$ units of labor. The N is determined by the free entry condition where $\theta W(t) = \mathbb{E}\left[V(t,Z)\right]$, an expected draw of Z from the equilibrium distribution.

Diffusion Costs, Market Clearing, and Aggregate Profits In order to upgrade its technology, a firm must hire ζ units of labor at real wage rate $W(t)$. Define S as the flow of agents choosing to upgrade, normalized by the number of varieties N. The total required labor for upgrades is $N\zeta S$ units of total labor. Given inelastic supply of 1 unit of labor, on a BGP,

$$1 = \underbrace{N\int_M^\infty \ell(Z)\mathrm{d}\Phi(Z)}_{\text{Variable Production}} + \underbrace{N\zeta S}_{\text{Upgrades}} + \underbrace{N\delta\theta}_{\text{Entry}} \tag{G.13}$$

As all costs are paid in labor and all final goods are all used for consumption, the consumer simply eats final goods output $C(t)$.

The aggregate real profits for the portfolio of all firm include variable profits and subtracts for real wages from the equity investment in upgrades,

$$\bar{\Pi} = N\int_M^\infty \Pi(Z)\mathrm{d}\Phi(Z) - N\zeta S W \tag{G.14}$$

Note that this does not include profits or losses from entering firms due to the zero-profit condition.

Firm's Dynamic Problem The firm maximizes the present discounted value of profits, using the discount rate, $r(t)$. Then with Z following an exogenously given Geometric Brownian motion with drift $\gamma + \sigma^2/2$ and variance σ^2,

$$r(t)V(t,Z) = \Pi(t,Z) + (\gamma + \sigma^2/2)Z\,\boldsymbol{\partial}_Z V(t,Z) + \frac{\sigma^2}{2}Z^2\,\boldsymbol{\partial}_{ZZ}V(t,Z) + \boldsymbol{\partial}_t V(t,Z) \tag{G.15}$$

$$V(t,M(t)) = \int_{M(t)}^\infty V(t,\tilde{Z})\mathrm{d}\Phi(t,\tilde{Z}) - \zeta W(t) \tag{G.16}$$

$$\boldsymbol{\partial}_Z V(t,M(t)) = 0 \tag{G.17}$$

Recall here that the discount rate, $r(t)$ has the death rate $\delta \geq 0$ built in through (G.3)

Free Entry Given the value of a new technology, similar to (G.16), and a cost of $\theta W(t)$, the free entry condition is,

$$\theta W(t) = \int_{M(t)}^\infty V(t,\tilde{Z})\mathrm{d}\Phi(t,\tilde{Z}) \tag{G.18}$$

Use (G.16),

$$(\theta - \zeta)W(t) = V(t,M(t)) \tag{G.19}$$

G.2 Balanced Growth Path

Define the following,

$$\tilde{\pi} \equiv \frac{1+\alpha-\varpi}{\alpha(\varpi-1)}(1/N - \zeta\alpha\left(g-\gamma-\alpha\tfrac{\sigma^2}{2}\right) - \delta\theta) \tag{G.20}$$

$$\nu \equiv \frac{4\tilde{\pi}\alpha - 2(\alpha-1)\varpi\left(\tilde{\pi}+\alpha\zeta(g-\gamma)\right) - 2\tilde{\pi} + (\alpha-1)\alpha\zeta(\varpi-1)^2\sigma^2 + 2(\alpha-1)\alpha\zeta(-\gamma+2g-r)}{(\alpha-1)\zeta\left(2(-\gamma\varpi+\gamma+g(\varpi-2)+r) - (\varpi-1)^2\sigma^2\right) - 2\tilde{\pi}} \tag{G.21}$$

$$a = \frac{\tilde{\pi}}{r-g-(\varpi-1)(\gamma-g+(\varpi-1)\sigma^2/2)} \tag{G.22}$$

Proposition 5 (Monopolistic Competition with Free Entry on a BGP)**.** *There exist a continuum of equilibria parameterized by α where,*

$$F(z) = 1 - e^{-\alpha z}. \tag{G.23}$$

The tail parameter of the underlying productivity distribution is α, while the tail parameter of the profit and firm size distributions, is given by

$$\hat{\alpha} \equiv (\varpi-1)\alpha. \tag{G.24}$$

Then, given the definitions for $\tilde{\pi}$ and ν, the equilibrium $\{g, N\}$ is a solution to the following system of two equations,

$$0 = -g + \frac{2\tilde{\pi}(\alpha-1)(\varpi-1)\sigma^2}{(\alpha-1)\zeta\left(2(-\gamma\varpi+\gamma+g(\varpi-2)+r) - (\varpi-1)^2\sigma^2\right) - 2\tilde{\pi}} + \alpha\sigma^2 + \gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} \tag{G.25}$$

$$\theta - \zeta = \frac{\tilde{\pi}(\nu+\varpi-1)}{\nu(-\gamma\varpi+\gamma+g(\varpi-2)+r) - \nu(\varpi-1)^2\sigma^2/2} \tag{G.26}$$

The value of a firm has the same structure of production in perpetuity plus the option value of adoption,

$$v(z) = ae^{(\varpi-1)z} + \frac{(\varpi-1)a}{\nu}e^{-\nu z} \tag{G.27}$$

Proof. The full equilibrium and specification is in Appendix G. □

To provide an illustrative example, consider an example with no drift or stochastic innovation, and no exogenous exit on the BGP (i.e., $\sigma = \Lambda = \gamma = \delta = 0$). Furthermore, choose $\varpi = 2$ to simplify the algebra. With this, given an α, the growth rate and number of varieties is,

$$g = \frac{r(\theta/\zeta - \alpha)}{(\alpha-1)^2} \tag{G.28}$$

$$N = \frac{(\alpha-1)^2}{r\alpha\zeta(1-2\alpha+\alpha\theta/\zeta)} \tag{G.29}$$

The key result is that growth rates are determined by the ratio θ/ζ. For this reason, a model with an exogenous number of firms, where the cost of adoption is interpreted as being relative to the cost of entry, delivers the same qualitative results as this model.

For a fixed cost ζ, an increasing θ acts as deterrent to entry, raising profits of incumbents and increasing growth rates. On the other hand, if $\delta > 0$, there would be a force acting in the opposite direction as more costly entry takes labor away from marginal production. The elasticity, ϖ, would also effect growth as it changes the relative value of entry.

From (G.24), higher markups lead to changes in the tail parameter of the size distribution, $\hat{\alpha}$, compared to the underlying productivity distribution. Therefore, when comparing growth rates to tail parameters of the firm size distribution in the data, it is important to adjust for the elasticity and markup. Define the markup as $\hat{\varpi} \equiv \varpi/(\varpi - 1) > 1$. Given an estimated $\hat{\alpha}$ from the firm size, profits, or revenue empirical distribution, the underlying tail index of the productivity distribution is

$$\alpha = (\varpi - 1)\hat{\alpha} = \frac{1}{\hat{\varpi} - 1}\hat{\alpha}. \tag{G.30}$$

This adjustment might explain some of the differences between the calibrated α in our model and those of the firm size distribution in the data. For example, with $\varpi = 3$ as discussed in Perla, Tonetti, and Waugh (2015), markups are 50%, and the adjustment factor is $(\varpi - 1) = 2$. From this, an underlying $\alpha = 2.12$ corresponds to a $\hat{\alpha} = 1.06$ tail parameter in the size or profits distribution—as used in Luttmer (2007). We will use equation (G.30) to give a rough conversion from the productivity distribution to those of the empirical revenue/size/profit distribution in the rest of the paper.[25]

G.3 Full Derivation

This section provides the explicit derivation of Appendix G.2

Static Conditions Follow a similar normalized approach to Appendix F.1 for $z \equiv \log(Z/M)$, and $F(t, z)$ using Main Paper (A.1). Normalize the real wage and aggregate consumption as,

$$w(t) \equiv \frac{W}{M} \tag{G.31}$$

$$c(t) \equiv \frac{C}{M} \tag{G.32}$$

For the value and real profits, further normalize to make them relative to the real normalized wage

$$\pi(t, z) \equiv \frac{\Pi(t, Z)}{w(t)M} \tag{G.33}$$

$$v(t, z) \equiv \frac{V(t, Z)}{w(t)M} \tag{G.34}$$

Use (G.10) and (G.31),

$$\tfrac{p(Z)}{P} = \hat{\varpi}\tfrac{w}{Z/M} = \hat{\varpi} w e^{-z} \tag{G.35}$$

Substitute into (G.6) and (G.11)

$$q(Z) = \hat{\varpi}^{-\varpi} w^{-\varpi} c \left(\tfrac{Z}{M}\right)^{\varpi} = \hat{\varpi}^{-\varpi} w^{-\varpi} c e^{\varpi z} \tag{G.36}$$

[25] When comparing to Luttmer (2007) and some other papers using monopolistic competition, keep in mind that the stochastic process in those papers was placed on profits or revenue directly rather than the underlying productivity distribution used here. Therefore, the tail parameters from those papers have something like (G.30) already built in, and require no adjustment.

Writing $(Z/M)^{\varpi} = e^{\varpi z}$ directly for simplicity, combine (G.6), (G.11) and (G.36),

$$q(Z) = \hat{\varpi}^{-\varpi} w^{-\varpi} c e^{\varpi z} \tag{G.37}$$

$$\ell(Z) = \hat{\varpi}^{-\varpi} w^{-\varpi} c e^{(\varpi-1)z} \tag{G.38}$$

Divide (G.7) by $P^{1-\varpi}$ and then substitute from (G.35) for $p(Z)/P$ to obtain

$$1 = N\hat{\varpi}^{1-\varpi} w^{1-\varpi} \int_M^{\infty} \left(\tfrac{Z}{M}\right)^{\varpi-1} \mathrm{d}\Phi(Z) \tag{G.39}$$

Simplify (G.39) by defining $\bar{Z}$, a measure of effective aggregate productivity. Then use an integral change of variables from Z to z to give normalized real wages in terms of parameters, $\bar{Z}$, and the productivity distribution

$$\bar{Z} \equiv N^{\frac{1}{\varpi-1}} \left[\int_0^{\infty} e^{(\varpi-1)z} \mathrm{d}F(z)\right]^{\frac{1}{\varpi-1}} = N^{\frac{1}{\varpi-1}} \mathbb{E}\left[e^{(\varpi-1)z}\right]^{\frac{1}{\varpi-1}} \tag{G.40}$$

$$w = \bar{Z}/\hat{\varpi} \tag{G.41}$$

Intuitively, this says that normalized real wages are proportional to normalized aggregate productivity, where the monopolistic friction lowers the wage share of aggregate output. Divide (G.12) and by Mw and substitute with (G.41) to obtain normalized profits,

$$\pi(Z) = \tfrac{1}{\varpi}\left(\tfrac{p(Z)}{P}\right)^{1-\varpi} \tfrac{c}{w} = \tfrac{1}{\varpi}\tfrac{c}{w}\bar{Z}^{1-\varpi} e^{(\varpi-1)z} \tag{G.42}$$

Divide (G.14) Mw and use (G.40) and (G.42) to find aggregate profits,

$$\bar{\pi} = N\tfrac{1}{\varpi}\tfrac{c}{w}\bar{Z}^{1-\varpi}\mathbb{E}\left[e^{(\varpi-1)z}\right] - N\zeta S \tag{G.43}$$

$$= \tfrac{1}{\varpi}\tfrac{c}{w} - N\zeta S \tag{G.44}$$

Combine (G.13), (G.38), (G.40) and (G.41) to obtain normalized aggregate labor demand

$$1 = \hat{\varpi}^{-\varpi} w^{-\varpi} c \bar{Z}^{\varpi-1} + N\zeta S + N\delta\theta \tag{G.45}$$

Multiply and divide the 2nd term of (G.45) by $w\hat{\varpi}$, then use (G.41),

$$1 = \frac{1}{\hat{\varpi}}\frac{c}{w}\hat{\varpi}^{1-\varpi}\bar{Z}^{\varpi-1} w^{1-\varpi} + N\zeta S + N\theta\delta \tag{G.46}$$

$$\frac{c}{w} = \hat{\varpi}(1 - N\zeta S - N\theta\delta) \tag{G.47}$$

Substitute (G.47) into (G.42), define a constant, and write the firm's profits (relative to wages)in terms of z parameters and the aggregates $\bar{Z}$ and S,

$$\pi(z) = \frac{1}{\varpi - 1}(1 - N\zeta S - N\theta\delta)\bar{Z}^{1-\varpi} e^{(\varpi-1)z} \tag{G.48}$$

Dynamic Normalization Given the equilibrium distribution and flow of adopters $S(t)$ and flow of entrants $\delta(t)$, define using (G.40) and (G.48) the following to decompose into the aggregate and idiosyncratic components of profits,

$$\tilde{\pi}(t) \equiv \frac{1/N(t) - \zeta S(t) - \theta\delta(t)}{(\varpi - 1)\mathbb{E}_t\left[e^{(\varpi-1)z}\right]} \tag{G.49}$$

$$\pi(t, z) = \tilde{\pi}(t) e^{(\varpi-1)z} \tag{G.50}$$

All of the general equilibrium conditions in the model have now been reduced to (G.41), (G.49) and (G.50). Normalize the value function by the normalized real wage and the scale,

$$v(t,z) \equiv \frac{V(t,Z)}{M(t)w(t)} \tag{G.51}$$

Or,

$$V(t,Z) = w(t)M(t)v(t,Z/M(t)) \tag{G.52}$$

The derivatives of (G.52) with respect to z are identical to those in Appendix F.1, but with the new multiplicative term. Differentiate the continuation value $V(t,Z)$ with respect to t in (G.52) and divide by $w(t)M(t)$, using the chain and product rule,

$$\frac{1}{w(t)M(t)}\boldsymbol{\partial}_t V(t,Z) = \frac{M'(t)}{M(t)}v(z,t) - \frac{M'(t)}{M(t)}\frac{Z}{M(t)}\boldsymbol{\partial}_z v(t,z) + \frac{M(t)}{M(t)}\boldsymbol{\partial}_t v(t,z) + \frac{w'(t)}{w(t)}v(z,t) \tag{G.53}$$

$$\frac{1}{M(t)}\boldsymbol{\partial}_Z V(t,Z) = \frac{1}{Z}\boldsymbol{\partial}_z v(t,z) \tag{G.54}$$

$$\frac{1}{M(t)}\boldsymbol{\partial}_{ZZ} V(t,Z) = \frac{1}{Z^2}\left(\boldsymbol{\partial}_{zz} v(t,z) - \boldsymbol{\partial}_z v(t,z)\right) \tag{G.55}$$

The growth rate is $g(t) \equiv M'(t)/M(t)$. Further define the growth rate of the relative wage, $g_w(t) \equiv w'(t)/w(t)$, which may be non-zero off of a BGP. Substitute these into (G.53), cancel out $M(t)$, and group $z = Z/M(t)$ to give

$$= (g(t) + g_w(t))v(z,t) - g(t)z\boldsymbol{\partial}_z v(t,z) + \boldsymbol{\partial}_t v(t,z) \tag{G.56}$$

Divide (G.15) by $M(t)w(t)$, use (F.15), (F.16), (G.48) to (G.50) and (G.53), and simplify,

$$(r(t) - g(t) - g_w(t))v(t,z) = \tilde{\pi}(t)e^{(\varpi-1)z} + (\gamma - g(t))\boldsymbol{\partial}_z v(t,z) + \frac{\sigma^2}{2}\boldsymbol{\partial}_{zz} v(t,z) \tag{G.57}$$

Further divide, (G.16) and (G.17) by $W(t)$,

$$v(t,0) = \int_0^\infty v(t,z)\mathrm{d}F(t,z) - \zeta \tag{G.58}$$

$$\boldsymbol{\partial}_z v(t,0) = 0 \tag{G.59}$$

The free entry condition from (G.19) normalizes to,

$$v(t,0) = \theta - \zeta \tag{G.60}$$

The stationary normalization of the KFE and distribution is identical to that in Appendix F.1. The exogenous death rate δ is simply added back in the same proportion of death, so has no effect on a BGP.

Stationary Equilibrium Equations In a stationary solution, $S(t)$ and $w(t)$ will be constant, so $g_w(t) = 0$. Furthermore, $\tilde{\pi}$ is constructed to be stationary. The complete set of equations for the stationary equilibrium is,

$$(r-g)v(z) = \tilde{\pi}e^{(\varpi-1)z} + (\gamma-g)v'(z) + \tfrac{\sigma^2}{2}v''(z) \tag{G.61}$$

$$v(0) = \int_0^\infty v(z)\mathrm{d}F(z) - \zeta \tag{G.62}$$

$$v'(0) = 0 \tag{G.63}$$

$$0 = (g-\gamma)F'(z) + \tfrac{\sigma^2}{2}F''(z) + SF(z) - S \tag{G.64}$$

$$F(0) = 0 \tag{G.65}$$

$$F(\infty) = 1 \tag{G.66}$$

$$\tilde{\pi} \equiv \frac{1/N - \zeta S - \delta\theta}{(\varpi-1)\mathbb{E}\left[e^{(\varpi-1)z}\right]} \tag{G.67}$$

$$r = \rho + g\Lambda + \delta \tag{G.68}$$

$$v(0) = \theta - \zeta \tag{G.69}$$

If–as will be shown–the equilibrium distribution is a power law with tail parameter $\alpha$and $1+\alpha > \varpi$, then,

$$\tilde{\pi} \equiv \frac{1+\alpha-\varpi}{\alpha(\varpi-1)}(1/N - \zeta S - \delta\theta) \tag{G.70}$$

$$S = \alpha\left(g - \gamma - \alpha\tfrac{\sigma^2}{2}\right) \tag{G.71}$$

Stationary Equilibrium Follow the techniques of Appendix F.2, and note that the KFE is identical. Hence,

$$F(z) = 1 - e^{-\alpha z} \tag{G.72}$$

Where α is related to S and g through (G.71). To solve (G.61) and (G.63), assume a solution of the form,

$$v(z) = ae^{(\varpi-1)z} + \frac{(\varpi-1)b}{\nu}e^{-\nu z} \tag{G.73}$$

Substitute this guess into (G.61) and (G.63) and equate undetermined coefficients to find that $a = b$, ν and a are given by

$$\nu = \frac{\gamma-g}{\sigma^2} + \sqrt{\left(\frac{g-\gamma}{\sigma^2}\right)^2 + \frac{r-g}{\sigma^2/2}} \tag{G.74}$$

$$a = \frac{\tilde{\pi}}{r-g-(\varpi-1)(\gamma-g+(\varpi-1)\sigma^2/2)} \tag{G.75}$$

Use (G.73) and (G.75) to find,

$$v(0) = \frac{\tilde{\pi}(\nu+\varpi-1)}{\nu(-\gamma\varpi+\gamma+g(\varpi-2)+r) - \nu(\varpi-1)^2\sigma^2/2} \tag{G.76}$$

Substitute (G.73), (G.75) and (G.76) into (G.62) to get an equation relating α and ν,

$$0 = \frac{2\tilde{\pi}(-\alpha(\varpi-2)+\nu+\varpi-1)}{(\varpi-1)^2\sigma^2 - 2(-\gamma\varpi+\gamma+g(\varpi-2)+r)} + (\alpha-1)\zeta(\alpha+\nu) \tag{G.77}$$

This can be solved further for ν

$$\nu = \frac{4\tilde{\pi}\alpha - 2(\alpha-1)\varpi\left(\tilde{\pi} + \alpha\zeta(g-\gamma)\right) - 2\tilde{\pi} + (\alpha-1)\alpha\zeta(\varpi-1)^2\sigma^2 + 2(\alpha-1)\alpha\zeta(-\gamma+2g-r)}{(\alpha-1)\zeta\left(2(-\gamma\varpi+\gamma+g(\varpi-2)+r) - (\varpi-1)^2\sigma^2\right) - 2\tilde{\pi}} \tag{G.78}$$

Equate (G.74) and (G.78)to find an expression in g and α

$$0 = -g + \frac{2\tilde{\pi}(\alpha-1)(\varpi-1)\sigma^2}{(\alpha-1)\zeta\left(2(-\gamma\varpi+\gamma+g(\varpi-2)+r) - (\varpi-1)^2\sigma^2\right) - 2\tilde{\pi}} + \alpha\sigma^2 + \gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g} \tag{G.79}$$

The free entry condition in (G.69) with (G.76) gives another expression

$$\theta - \zeta = \frac{\tilde{\pi}(\nu+\varpi-1)}{\nu(-\gamma\varpi+\gamma+g(\varpi-2)+r) - \nu(\varpi-1)^2\sigma^2/2} \tag{G.80}$$

Given a fixed α and the substitutions for ν, r, and $\tilde{\pi}$ from (G.68), (G.70) and (G.78), a solution is an N and g fulfilling (G.79) and (G.80).

Example: Exogenous $N = 1$

Substitute with (G.70), (G.71) and (G.79) to find an implicit equation in g given the α. The set of $\{\alpha, g\}$ fulfilling this equation is the set of admissible stationary solutions.

$$0 = \zeta + \frac{\left(\alpha\zeta\left(\alpha\sigma^2 + 2\gamma - 2g\right) + 2\right)\left(\gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} - g + (\varpi-1)\sigma^2\right)}{\alpha\left(\alpha\sigma^2 + \gamma + \sqrt{(g-\gamma)^2 + 2\sigma^2(r-g)} - g\right)\left((\varpi-1)^2\sigma^2 - 2(-\gamma\varpi+\gamma+g(\varpi-2)+r)\right)} \tag{G.81}$$

In case of $\sigma = 0$ with an arbitrary γ and ϖ, the solution is,

$$g = \frac{1 - \alpha(\rho - \gamma(1+\alpha))\zeta}{\alpha(\Lambda+\alpha)\zeta} \tag{G.82}$$

And in the baseline example with $\sigma = 0$ and no risk aversion or drift, $\Lambda = \gamma = 0$,

$$g = \frac{1 - \alpha r\zeta}{\alpha^2\zeta} \tag{G.83}$$

Note the independence of ϖ in the growth rates of (G.82) and (G.83). However, the stationary distribution of profits, $\pi(z)$, still depends on the ϖ elasticity. Doing a change of variables with (G.48), where $\pi(z) \propto e^{(\varpi-1)z}$: if the z distribution is exponential with "tail" parameter α (i.e., exponential since $\log Z$, which translates to a power-law in Z), then the distribution of profits or size is Pareto with tail parameter $\alpha/(\varpi-1)$ through (G.48).

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